THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

A BAYES SOLUTION FOR THE PROBLEM OF RANKING POISSON PARAMETERS

by

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Technical Report No. 57
Department of Statistics THEMIS Contract

March 13, 1970

Research sponsored by the Office of Naval Research
Contract N00014-68-A-0515
Project NR 042-260

DEPARTMENT OF STATISTICS
Southern Methodist University

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Abstract

The Poisson distribution is used as the mathematical model for the random behavior of many natural phenomena. For example, it is often used in reliability studies for the number of failures in some fixed time interval.

The Poisson distribution (or population) is a one-parameter family of distributions, the parameter being the mean value. This paper is concerned with the problem of ranking the parameters for n different Poisson populations based on fixed equal size samples from each population. A Bayes solution is derived for several types of loss functions and gamma priors, under the usual assumptions of symmetric and additive losses and symmetric priors.

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Summary. It is desired to rank the parameters from a set of Poisson populations based on fixed, equal size samples from each population. A Bayes solution is derived for several types of loss functions and gamma priors, under the usual assumptions of symmetric and additive losses and symmetric priors.

1. Introduction. We assume that a set of n Poisson populations have been observed, yielding (through reduction by sufficiency) the data $X_1, \ldots, X_n$. (The results presented apply equally as well to a set of Poisson processes observed over time periods of equal length.) The object is to determine an ordering of the parameters $\lambda_1, \ldots, \lambda_n$, allowing the possibility of "ties". That is, for each pair of parameters λ_i, λ_j, there are three decisions available: λ_i is greater than λ_j, λ_j is greater than λ_i, λ_i and λ_j are unranked. The approach we adopt is thus closely related to that of earlier work by Duncan [2], [3], Bland and Bratcher [1], and Naik [5].

2. General description and solution of the problem. We make the following basic assumptions:

(i) The prior information, in the form of a probability distribution, is the same for each λ_i. We denote the prior by $\varphi(\lambda)$.

(ii) The loss function for the overall problem can be taken to be the sum of the loss functions for certain component problems.

* This research was sponsered by the office of Naval Research, Contract No. N000 14-68-A-0515, Project No. NR 042-260.

(iii) Finally, the loss functions for the component problems are symmetric in the sense that permutations of the decisions and corresponding parameters leave the loss unchanged.

The assumptions (i) and (iii) can best be summarized and justified by saying that the populations are to be treated equally <u>a priori</u> and the seriousness of an incorrect decision is the same regardless of the populations concerned. The concept of additive loss, which appears in assumption (ii), was introduced and discussed by Lehmann [4].

The component problems mentioned above are two-decision problems which are easily solved. In particular, for each pair of parameters, say λ and μ, there are two component two-decision problems related to the three-decision problem described in section 1. One consists of the decisions d_0: λ and μ are unranked and d_1: $\lambda > \mu$. The other problem is obtained by interchanging the parameters. Denote by $L(d;\lambda,\mu)$ the loss incurred in making decision d when λ and μ are the true values of the parameters.

It is seen that a solution to each of the component two-decision problems for λ and μ will specify a solution to the three-decision problem, provided only that the solutions to the component problems are compatible. That is, we require that it is never possible to make the two decisions $\lambda > \mu$ and $\mu > \lambda$ simultaneously. In the same way, solutions to the three-decision problems lead to a solution for the overall problem, if inconsistent decisions (such as $\lambda_1 > \lambda_2$, $\lambda_2 > \lambda_3$, and $\lambda_3 > \lambda_1$) are never made. We do, however, allow decisions of the type: λ_1 and λ_2 are unranked, λ_2 and λ_3 are unranked, but $\lambda_1 > \lambda_3$.

Because of the assumptions of symmetric losses and identical priors only one two-decision problem actually need be solved; these assumptions are not essential and the problem could be solved without them.

<u>Theorem:</u> Under assumptions (i), (ii), and (iii), the Bayes rule for the overall multiple comparisons problem is given by the following: for each pair λ_i, λ_j, make decision d_1 if $h(x_i,x_j) > 0$; otherwise, make decision d_0, where

$$h(x_i,x_j) = \int_0^\infty\int_0^\infty [L(d_0;\lambda_i,\lambda_j)-L(d_1;\lambda_i,\lambda_j)]f(x_i|\lambda_i)f(x_j|\lambda_j)\varphi(\lambda_i)\varphi(\lambda_j)d\lambda_i d\lambda_j,$$

provided these component solutions are compatible. ($f(x|\lambda)$ is the probability mass function of the observed Poisson variables.) Note that the solution is unique if, and only if, the probability that $h(x_i, x_j) = 0$ is zero.

We omit the proof; it is tedious algebraically and is available for similar situations, along with further discussion and background, in the references cited above.

3. <u>The Bayes rule for gamma priors and absolute error loss.</u> Suppose that we observe X and Y from Poisson populations with parameters λ and μ, respectively. We consider first the loss function given by

$$L(d_0;\lambda,\mu) = \begin{cases} 0 & , \text{ if } \lambda \le \mu \\ k_0(\lambda-\mu), & \text{if } \lambda > \mu \end{cases}$$

$$L(d_1;\lambda,\mu) = \begin{cases} k_1(\mu-\lambda), & \text{if } \lambda \le \mu \\ 0 & , \text{ if } \lambda > \mu \end{cases} \tag{1}$$

where $k_1 \ge k_0 > 0$ are known, and the prior density function is

$$\varphi(\lambda) = \begin{cases} \beta^{\alpha+1}\lambda^{\alpha}e^{-\beta\lambda}/\Gamma(\alpha+1), & \text{if } \lambda > 0 \, (\alpha > -1, \ \beta > 0) \\ 0 & , \text{ if } \lambda \le 0 \end{cases}$$

where α and β are assumed known.

Then, writing $(u)^+$ for the positive part of u,

$$h(x,y) = \int_0^\infty\int_0^\infty [k_0(\lambda-\mu)^+ - k_1(\mu-\lambda)^+][x!y!\Gamma(\alpha+1)\Gamma(\alpha+1)]^{-1}\lambda^{x+\alpha}\mu^{y+\alpha}$$
$$\exp[-(1+\beta)(\lambda+\mu)]d\lambda d\mu.$$

Since we are interested only in the sign of $h(x,y)$, we can freely neglect positive multiples, even those involving x and y. We will use the symbol "$\propto$" in this sense only. We introduce the variable $r = \mu/\lambda$ and let $k = k_1/k_0$. Then

$$h(x,y) \propto \int_0^\infty\int_0^\infty [\lambda(1-r)^+ - k\lambda(r-1)^+]\lambda^{x+y+2\alpha}r^{y+\alpha}\exp[-\lambda(1+\beta)(1+r)]\lambda d\lambda dr$$

$$= \int_0^\infty \left[(1-r)^+ - k(r-1)^+\right]r^{y+\alpha}\int_0^\infty \lambda^{x+y+2\alpha+2}\exp\left[-\lambda(1+\beta)(1+r)\right]d\lambda dr$$

$$\propto \int_0^\infty \left[(1-r)^+ - k(r-1)^+\right]r^{y+\alpha}(1+r)^{-(x+y+2\alpha+3)}dr$$

$$= \int_0^1 (1-r)r^{y+\alpha}(1+r)^{-(x+y+2\alpha+3)}dr - k\int_1^\infty (r-1)r^{y+\alpha}(1+r)^{-(x+y+2\alpha+3)}dr.$$

Changing from r to 1/r in the second integral and then to $z = r/(1+r)$ in both, we get

$$h(x,y) \propto \int_0^{1/2}(1-2z)z^{y+\alpha}(1-z)^{x+\alpha}dz - k\int_0^{1/2}(1-2z)z^{x+\alpha}(1-z)^{y+\alpha}dz. \qquad (2)$$

Now let $I_{1/2}(m,n) = \Gamma(m+n+2)\left[\Gamma(m+1)\Gamma(n+1)\right]^{-1}\int_0^{1/2}z^m(1-z)^n dz$, the incomplete beta integral. Making use of the identity $I_{1/2}(m,n) = 1-I_{1/2}(n,m)$, we find that

$$h(x,y) \propto x-y-(k+1)(x+y+2\alpha+2)I_{1/2}(x+\alpha,y+\alpha) + 2(y+\alpha+1)I_{1/2}(x+\alpha,y+\alpha+1)$$
$$+ 2k(x+\alpha+1)I_{1/2}(x+\alpha+1,y+\alpha).$$

Finally, $I_{1/2}(m,n) = I_{1/2}(m-1,n+1) - \Gamma(m+n+2)\left[\Gamma(m-1)\Gamma(n+2)\right]^{-1}2^{-(m+n+1)}$, so

$$h(x,y) \propto x-y-k\Gamma(x+y+2\alpha+3)\left[\Gamma(x+\alpha+1)\Gamma(y+\alpha+2)2^{x+y+2\alpha+1}\right]^{-1}$$
$$+2\left[kx+y+(k+1)(\alpha+1)\right]I_{1/2}(x+\alpha,y+\alpha+1)$$
$$-(k+1)(x+y+2\alpha+2)I_{1/2}(x+\alpha,y+\alpha).$$

It remains to show that the component solutions are compatible. We can write (2) as

$$\int_0^{1/2} (1-2z)z^{x+\alpha}(1-z)^{x+\alpha}\left[z^{y-x}-k(1-z)^{y-x}\right]dz$$

$$= \int_0^{1/2} (1-2z)z^{y+\alpha}(1-z)^{x+\alpha}\left[1-k\ \left(\frac{1-z}{z}\right)^{y-x}\right]dz$$

The last factor in the integrand is clearly non-positive for $x \le y$ (since k is assumed not less than one) whereas the other factors are positive. So for $x \le y$, $h(x,y) \le 0$ and decision d_0 is made. That is, the set $\{(x,y):h(x,y) > 0\}$ is contained in the set $\{(x,y): x > y\}$. In the other component problem involving λ and μ, the decision $\mu > \lambda$ would be made when $h(y,x) > 0$. But by symmetry, the set $\{(x,y):h(x,y) > 0\}$ is a subset of $\{(x,y):y > x\}$. Hence the sets $\{(x,y):h(x,y) > 0\}$ and $\{(x,y):h(y,x) > 0\}$ do not intersect, so the solutions given by the component two-decision problems are compatible. The compatibility condition is seen to be that the ratio k_1/k_0 is not less than unity.

4. <u>Other loss functions.</u> If $|\lambda-\mu|$ is replaced in (1) by $|\lambda-\mu|^p$, for any non-negative integer p, the same sequence of steps gives

$$h(x,y) \propto \sum_{j=0}^{p} \binom{p}{j}(-2)^j\left[\Gamma(x+y+2\alpha+j+2)\right]^{-1}\left[\Gamma(x+y+1)\Gamma(y+\alpha+j+1)\right.$$
$$\left. I_{1/2}(y+\alpha+j,x+\alpha)-k\Gamma(x+\alpha+j+1)\Gamma(y+\alpha+1)I_{1/2}(x+\alpha+j,y+\alpha)\right] \quad (3)$$

Another form of loss function is obtained by substituting $(\lambda/\mu)^\gamma$ for $(\lambda-\mu)$ and $(\mu/\lambda)^\gamma$ for $(\mu-\lambda)$ for any $0 \le \gamma \le \alpha+1$. Then is is easily verified that

$$h(x,y) \propto \Gamma(x+\alpha+\gamma+1)\Gamma(y+\alpha-\gamma+1)I_{1/2}(y+\alpha-\gamma,x+\alpha+\gamma)$$
$$-k\Gamma(x+\alpha-\gamma+1)\Gamma(y+\alpha+\gamma+1)I_{1/2}(x+\alpha-\gamma,y+\alpha+\gamma). \quad (4)$$

In the special case of (3) with $p = 0$ or of (4) with $\gamma = 0$ we get

$$h(x,y) \propto 1 - (k+1)\, I_{1/2}(x+\alpha, y+\alpha).$$

The same compatibility condition applies in these cases.

5. <u>Form of the critical region.</u> It is of interest to verify that for fixed y, we make decision d_0 for $x \leq x_0(y)$ and decision d_1 otherwise. This result is not obvious since $h(x,y)$ is not monotonic in x for fixed y; the result is true, however, for the cases considered in the previous section, as can be established by the following reasoning. The form of equation (2) corresponding to the loss function involving $(\lambda/\mu)^{\gamma}$ and $(\mu/\lambda)^{\gamma}$ is

$$\int_0^{1/2} z^{y+\alpha-\gamma}(1-z)^{y+\alpha+\gamma}\left[(1-z)^{x-y} - kz^{x-y}\right]dz.$$

Suppose that for some given x and y this expression, a positive multiple of $h(x,y)$, is negative. Then $h(x,y) < 0$ and $h(x-1,y)$ is a positive multiple of

$$\int_0^{1/2} z^{y+\alpha-\gamma}(1-z)^{y+\alpha+\gamma}\left[(1-z)^{x-y-1} - kz^{x-y-1}\right]dz$$

$$\leq \int_0^{1/2} z^{y+\alpha-\gamma}(1-z)^{y+\alpha+\gamma}\left[2(1-z)^{x-y} - 2kz^{x-y}\right]dz$$

$$= 2\int_0^{1/2} z^{y+\alpha-\gamma}(1-z)^{y+\alpha+\gamma}\left[(1-z)^{x-y} - kz^{x-y}\right]dz.$$

Therefore, $h(x-1,y)$ is negative whenever $h(x,y)$ is negative. An argument along the same lines shows that for the loss function involving a power of the absolute difference of the parameters the same statement holds. Furthermore, for either type of loss, $h(x,y)$ positive implies that $h(x+1,y)$ is positive. It follows that for fixed y, $h(x,y)$ changes sign at most once.

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UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (*Corporate author*) SOUTHERN METHODIST UNIVERSITY	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP UNCLASSIFIED

3. REPORT TITLE

"A Bayes solution for the problem of ranking Poisson parameters"

4. DESCRIPTIVE NOTES (*Type of report and inclusive dates*)

Technical Report

5. AUTHOR(S) (*First name, middle initial, last name*)

D. O. Dixon

R. P. Bland

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
March 13, 1970	8	5

8a. CONTRACT OR GRANT NO. N00014-68-A-0515	9a. ORIGINATOR'S REPORT NUMBER(S) 57
b. PROJECT NO. NR042-260	
c.	9b. OTHER REPORT NO(S) (*Any other numbers that may be assigned this report*)
d.	

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Office of Naval Research

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DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

UNCLASSIFIED
Security Classification

A-31408