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URBAN CASUALTY ESTIMATION AND THE CIRCULAR COVERAGE FUNCTION. II.

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1.

In an earlier paper [6] a city distributed uniformly over a circle was considered in a model for an n-weapon attack using the circular normal damage function and aiming error developed by Hunter [4]. The expected number of casualties was evaluated using the Circular Coverage Function $p(R,r)$ ([1], [2]).

Many damage curves do not fit the single-or two-term circular normal form of Hunter, or the elliptical normal form discussed by Grubbs [3]. These "cookie-cutter" curves are better fitted by the damage function

$$P(r) = \begin{cases} 1 & , \ 0<r\leq k \\ Q\exp(-cr^2), & r>k,\ Q\geq 0, c>0, \end{cases} \tag{1.1}$$

where r is the distance from ground zero.

We consider a single-weapon attack on a city having a circular normal distribution of population about the origin (see Sherratt [7]); the aiming error, as in Hunter's model [4], is assumed to be circular normal about the point of impact (x_1, y_1), and the damage function of the form (1.1). The Circular Coverage Function, discussed in [6], will again figure in the evaluation of the expected number of casualties.

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Section 2 evaluates the expected casualties when the impact point is given, and in Section 3 an expression for the damage function, modified to include aiming error, is obtained. Section 4 estimates the expected casualties when the aiming point (X_1, Y_1) is given. To the writer's knowledge, no previous model has led to such a result for the cookie-cutter damage function (1.1).

2. Single weapon blast at (x_1, y_1).

Let the density of city population be (as in Sherratt [7])

$$\rho(x,y) = P_0 \exp[-(x^2+y^2)/2b^2]. \tag{2.1}$$

Consider a weapon blast at (x_1, y_1), and let T_1 denote the expected casualties. Then, following Fig. 1,

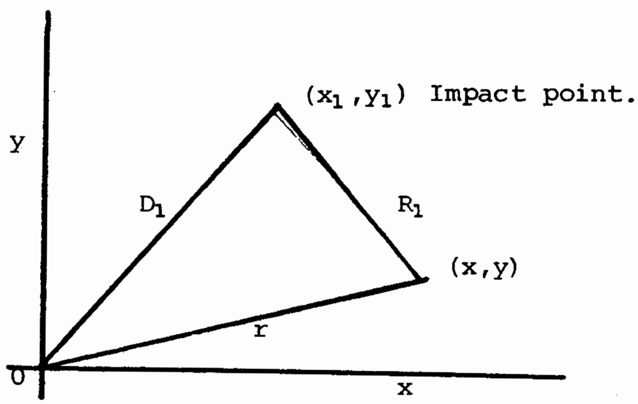

Fig. 1

$$T_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x,y)P(R_1)\,dxdy \tag{2.2}$$

If $\quad S = \left\{(x,y)\,\middle|\,(x-x_1)^2 + (y-y_1)^2 \le k^2\right\}$

and $\quad S' = \left\{(x,y)\,\middle|\,(x,y)\notin S\right\}$

then

$$T_1 = \iint_S P_0 \exp\left[-\frac{x^2+y^2}{2b^2}\right] dxdy + \iint_{S'} P_0 Q \exp\left[-\frac{x^2+y^2}{2b^2}\right.$$

$$\left. - c\left\{(x-x_1)^2 + (y-y_1)^2\right\}\right] dxdy$$

$$= P_0 I_1 + P_0 Q I_2, \tag{2.3}$$

where

$$I_1 = \iint_S \exp[-(x^2+y^2)/2b^2]\,dxdy,$$

and

$$I_2 = \iint_{S'} \exp[-(x^2+y^2)/2b^2 - c\left\{(x-x_1)^2 + (y-y_1)^2\right\}]\,dxdy$$

Let

$$d_1^2 = x_1^2 + y_1^2 \quad , d_1>0$$

and

$$\gamma^2 = 1+2b_2 c \quad , \gamma>0$$

In order to evaluate I_1 and I_2, we require the Circular Coverage Function ([2], [6])

$$p(R,r) = 1/(2\pi)\iint_R \exp\left\{-\tfrac{1}{2}(x^2+y^2)\right\}dxdy$$

integrated over the region R in which

$$(x-a')^2 + (y-b')^2 \leq R^2,$$

$$a'^2 + b'^2 = r^2.$$

Then

$$I_1 = 2\pi b^2 p(k/b, d_1/b). \tag{2.4}$$

We also require Lemma 1 and Corollary 1 of [6].

The corollary proved there states (in the present notation):

$$\int_{S'} \exp[-Ax^2+2Bx-Ay^2+2By]dxdy = \pi A^{-1}\exp[(B^2+B'^2)A]$$

$$\cdot\left\{1-p[(2A)^{\frac{1}{2}}k,\ (2(B^2+B'^2)/A)]\right\} \tag{2.5}$$

where $A>0$, and B,B' are arbitrary.

Then, setting $u = x-x_1$, $v=y-y_1$,

$$I_2 = \exp[-d_1^2/(2b^2)]\cdot\iint_{S'} \exp[-(u^2+v^2)\gamma^2/(2b^2)$$

$$-x_1u/b^2-y_1v/b^2]\,dudv$$

$$= (2\pi b^2/\gamma^2)\exp(-cd_1^2/\gamma^2)\left\{1-p[k\gamma/b, d_1/(\gamma b)]\right\} \tag{2.6}$$

on applying Corollary 1 of [6] over the circle S: $u^2+v^2\leq k^2$, and S', and simplifying.

Hence, applying (2.4) and (2.5) in (2.3), the conditional expected casualties, given impact occurs at (x_1, y_1), are given by

$$T_1 = (2\pi P_0 Q b^2/\gamma^2)\exp(-cd_1^2/\gamma^2)\left\{1-p(k\gamma/b, d_1/(b\gamma))\right\}$$
$$+ 2\pi P_0 b^2 p(k/b, d_1/b) \qquad (2.7)$$

3. Modified Damage Function.

If aiming error about an aiming point (X,Y) is considered, it is convenient to obtain an expression which modifies the damage function. Fig. 2 shows the model in diagrammatic form.

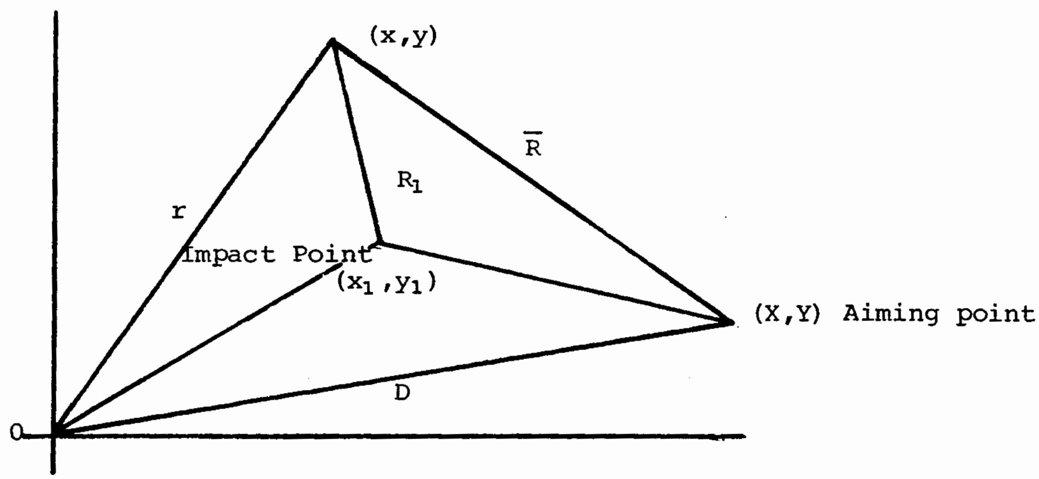

Fig. 2

Let the distance from the aiming point to (x,y) be $\overline{R}$, and let $\alpha^2 = 1 + 2\sigma^2 c$,

where σ^2 is the variance parameter of the circular normal distribution of aiming error about (X,Y). Then we obtain the modified damage function in the form of

Theorem 1. If the probability of an individual becoming casualty at a distance r from the impact point of a weapon is given by (1.1), and if the impact point has a circular normal distribution about the aiming point with variance parameter σ^2, then the probability of an individual at (x,y) becoming casualty is given by

$$P_{(x,y)}(\mathrm{Cas}) = p[k/\sigma, \bar{R}/\sigma] + (Q/\alpha^2)\exp(-c\bar{R}^2/\alpha^2)\left\{1-p(k\alpha/\sigma,\ \bar{R}/(\alpha\sigma))\right\} \tag{3.1}$$

Proof: Let $S_{(x,y)} = \left\{(x,y) \mid (x-x_1)^2 + y-y_1{}^2) \le k^2\right\}$

where (x_1, y_1) is the impact point; thus $S_{(x,y)}$ is the set of (x,y) such that the probability of becoming casualty is 1, when (x_1, y_1) is specified. Then following Hunter's argument (cf. his Eq.(10)),

$$P_{(x,y)}(\hat{\mathrm{Cas}}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_{(x,y)}(\mathrm{Cas} \mid \text{impact in element } dx_1 dy_1)$$

$$\cdot \Pr(\text{impact in element } dx_1 dy_1)\, dx_1 dy_1$$

$$= \iint_{S_{(x,y)}} (1/(2\pi\sigma^2))\exp\left[-(1/(2\sigma^2))\left\{(x_1-X)^2+(y_1-Y)^2\right\}\right]dx_1 dy_1$$

$$+\iint_{S'_{(x,y)}} Q\exp[-c(x-x_1)^2-c(y-y_1)^2].\ \exp\left[-(1/(2\sigma^2))\left\{(x_1-X)^2\right.\right.$$

$$\left.\left. +\ y_1-Y)^2\right\}\right]dx_1 dy_1$$

$$= I_1^* + QI_2^* \quad ,\text{say}. \tag{3.2}$$

Again, put $u=x-x_1$, $v=y-y_1$. Then

$$I_1^* = \iint_S (2\pi\sigma^2)^{-1}\exp\left[-(2\sigma^2)^{-1}\left\{(u+x-X)^2+(v+y-Y)^2\right\}\right]dudv$$

$$=(2\pi\sigma^2)^{-1}\exp[-(2\sigma^2)^{-1}\bar{R}^2]\iint_S \exp\left[-(2\sigma^2)^{-1}\left\{u^2+v^2-2u(x-X)\right.\right.$$

$$\left.\left.-2v(y-Y)\right\}\right]dudv$$

$$=(2\pi\sigma^2)^{-1}\exp[-(2\sigma^2)^{-1}\bar{R}^2].2\pi\sigma^2\exp\left[(2\sigma^2)^{-1}\bar{R}^2\right]p\left[k/\sigma,\bar{R}/\sigma\right]$$

$$= p(k/\sigma,\bar{R}/\sigma).$$

Likewise

$$I_2^* = (2\pi\sigma^2)^{-1}\exp(-\bar{R}^2/(2\sigma^2))\iint_{S'}\exp\left[\left\{-\alpha^2(u^2+v^2)\right.\right.$$

$$\left.\left.-2(x-X)u-2(y-Y)v\right\}/(2\sigma^2)\right]dudv$$

$$= (1/\alpha^2)\exp(-c\bar{R}^2/\alpha^2)\left\{1-p(k\alpha/\sigma,\ \bar{R}(\sigma\alpha)^{-1})\right\}$$

on applying (2.5) and simplifying.

The result follows, from (3.2). Q.E.D.

Aiming error is now incorporated into the damage function by Theorem 1, and casualties from a single weapon attack may now be estimated.

4. Single Weapon Attack with Aiming Error.

Denote by P(Cas) the damage function (3.1). If $\rho(x,y)$ is the city population density at (x,y), then the expected number of casualties C from a single weapon aimed at (X,Y) is given by

$$C = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x,y)\,P(\mathrm{Cas})\,dxdy$$

Using the circular normal urban density model (2.1), let

$$\beta^2 = \alpha^2 + 2b^2c, \quad \beta>0.$$

Then Theorem 1 gives

$$C = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_0\exp[-(2b^2)^{-1}(x^2+y^2)]\left[p(k/\sigma,\bar{R}/\sigma)+Q\alpha^{-2}\exp(-c\bar{R}^2/\alpha^2)\right.$$
$$\left.\cdot\left\{1-p(k\alpha/\sigma,\bar{R}(\sigma\alpha)^{-1})\right\}\right]dxdy \tag{4.1}$$

$$= P_0 I_3 + (P_0 Q/\alpha^2)(I_4-I_5) \tag{4.2}$$

where

$$I_3 = \int_{-\infty}^{\infty} \exp[-(x^2+y^2)/(2b^2)]\,p(k/\sigma,\bar{R}/\sigma)\,dxdy$$

$$I_4 = \int_{-\infty}^{\infty} \exp[-(x^2+y^2)/(2b^2)-c\left\{(x-X)^2+(y-Y)^2\right\}/\alpha^2]\,dxdy$$

$$I_5 = \int_{-\infty}^{\infty} \exp[-(x^2+y^2)/(2b^2)-c\bar{R}^2/\alpha^2]\,p(k\alpha/\sigma,\bar{R}(\sigma\alpha)^{-1})\,dxdy$$

First,

$$I_4 = \exp[-(X^2+Y^2)c/\alpha^2]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-\beta^2(x^2+y^2)/(2\alpha^2b^2)+2cXx/\alpha^2$$
$$+2cYy/\alpha^2]\,dxdy$$

$$=2(\alpha^2/\beta^2)\pi b^2 \exp[-cD^2/\alpha^2+2b^2c^2D^2/(\alpha^2\beta^2)] \tag{4.3}$$

using (2.5) of [6], and where

$$D^2 = X^2+Y^2 \text{ as in Fig. 2.}$$

In order to evaluate I_3 and I_5, we require a Bessel function, and then to apply a property of $p(R,r)$. First, we rotate the axes so that (X,Y) lies on the new x'-axis, and then translate the origin to (X,Y). Thus let

$$x = (x'+D)\cos\alpha' - y'\sin\alpha'$$

$$y = y'\cos\alpha' + (x'+D)\sin\alpha'$$

where $\tan\alpha' = Y/X$.

$\bar{R}$ is invariant under translation and rotation, and $\bar{R}^2 = x'^2+y'^2$.

So $\quad I_3 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[(-1/(2b^2))(x'^2+y'^2+2x'D+D^2)]\,p(k/\sigma,\bar{R}/\sigma)\,dx'dy'$

Finally, introduce polar coordinates,

$$x' = \bar{R}\cos\theta$$

$$y' = \bar{R}\sin\theta.$$

Then

$$I_3 = \int_0^{\infty}\int_0^{2\pi} \exp[-(\bar{R}^2+D^2)/(2b^2)-D\bar{R}\cos\theta/b^2]\,p(k/\sigma,\bar{R}/\sigma)\,\bar{R}\,d\theta\,d\bar{R}$$

$$= 2\pi\int_0^{\infty}\exp[-(\bar{R}^2+D^2)/(2b^2)]\,I_0(D\bar{R}/b^2)\,p(k/\sigma,\bar{R}/\sigma)\,\bar{R}\,d\bar{R}, \tag{4.4}$$

where

$$I_0(z) = (2\pi)^{-1}\int_0^{2\pi} \exp(-z\cos\theta)\,d\theta$$

is the modified zero-order Bessel function of the first kind.

We require the result

$$p(r_0/s,d/s) = \sigma^{-2}\int_0^{\infty} p(r_0/\delta,r/\delta)\exp[-(r^2+d^2)/(2\sigma^2)]\,I_0(rd/\sigma^2)\,r\,dr$$

where (4.5)

$$s^2 = \sigma^2+\delta^2$$

(See, for example, Egs. (15) and (16) of [5]).

Then it immediately follows from (4.4) and (4.5) that

$$I_3 = 2\pi b^2 p[k(b^2+\sigma^2)^{-\frac{1}{2}}, D(b^2+\sigma^2)^{-\frac{1}{2}}] \tag{4.6}$$

Next, we make the same translation and rotation for I_5 as for I_3, and similarly transform to polar coordinates. We get

$$I_5 = 2\pi\int_0^\infty p[k\alpha/\sigma, \bar{R}/(\sigma\alpha)]\exp[-(\bar{R}^2+D^2)/(2b^2)-c\bar{R}^2/\alpha^2]$$
$$.I_0(D\bar{R}/b^2)\bar{R}d\bar{R}$$

In order to apply (4.5) to this integral, it is necessary to rearrange with the use of auxiliary factors. Writing I_5 in the form

$$I_5 = 2\pi\exp(-cD^2/\beta^2)\int_0^\infty p\left[k\alpha^2/\sigma\alpha), \bar{R}/(\sigma\alpha)\right]$$
$$.\exp\left[-\left\{\bar{R}^2+(D\alpha^2/\beta^2)^2\right\}/(2b^2\alpha^2/\beta^2)\right].I_0\left[D(\alpha^2/\beta^2)\bar{R}/(b\alpha/\beta)^2\right]\bar{R}d\bar{R},$$

We can apply (4.5) to get

$$I_5 = 2\pi(\alpha^2/\beta^2)\exp(-cD^2/\beta^2)p[k\alpha(\sigma^2+b^2/\beta^2)^{-\frac{1}{2}}, D\alpha\beta^{-2}(\sigma^2+b^2/\beta^2)^{-\frac{1}{2}}] \tag{4.7}$$

Collecting (4.3), (4.6) and (4.7), C is obtained. We state this in the form of

<u>Theorem 2</u>. Under the model described (viz. urban population density (4.1), damage function (1.1), aiming error as in Theorem 1), then the expected number of casualties from a single weapon aimed at (X,Y) is

$$C = 2\pi P_0 b^2\Big[p(k/(b^2+\sigma^2)^{\frac{1}{2}}, D/(b^2+\sigma^2)^{\frac{1}{2}}) + (Q/\beta^2)\exp(-cD^2/\beta^2)$$
$$.\left\{1-p(k\alpha\beta/(b^2+\sigma^2\beta^2)^{\frac{1}{2}}, D(\alpha/\beta)/(b^2+\sigma^2\beta^2)^{\frac{1}{2}})\right\}\Big] \tag{4.8}$$

where $\alpha^2 = 1+2c\sigma^2, \quad \alpha>0$

$\beta^2 = \alpha^2+2b^2c, \quad \beta>0.$

Proof: This follows from (4.2), (4.3), (4.6) and (4.7). Q.E.D.

If $S_0{}^2 = b^2+\sigma^2, \quad S_0>0,$

$S_1{}^2 = \alpha^2+2b^2c, \quad S_1>0,$

then (4.8) can be written

$$C = 2\pi P_0 b^2 \left[p(k/S_0, D/S_0) + Q\beta^{-2} \exp(-cD^2/\beta^2) \cdot \left\{ 1 - p(k\alpha\beta/S_1, D\alpha/(\beta S_1)) \right\} \right]. \tag{4.9}$$

The proportion of the city population which falls casualty to this single weapon attack is $C/(2\pi P_0 b^2)$.

The programming of (4.9) can be done by Wegner's approximations [8] to $p(R,r)$, and these are set out in the author's earlier paper [6].

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