Asymptotic expansions and hazard rates for compound and first-passage distributions

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A general theory which provides asymptotic tail expansions for density, survival, and hazard rate functions is developed for both absolutely continuous and integer-valued distributions. The expansions make use of Tauberian theorems which apply to moment generating functions (MGFs) with boundary singularities that are of gamma-type or log-type. Standard Tauberian theorems from Feller (1971) can provide a limited theory but these theorems do not suffice in providing a complete theory as they are not capable of explaining tail behaviour for compound distributions and other complicated distributions which arise in stochastic modelling settings. Obtaining such a complete theory for absolutely continuous distributions requires introducing new "Ikehara" conditions based upon Tauberian theorems whose development and application have been largely confined to analytic number theory. For integer-valued distributions, a complete theory is developed by applying Darboux's theorem used in analytic combinatorics. Characterizations of asymptotic hazard rates for both absolutely continuous and integer-valued distributions are developed in conjunction with these expansions. The main applications include the ruin distribution in the Cramér-Lundberg and Sparre Andersen models, more general classes of compound distributions, and first-passage distributions in finite-state semi-Markov processes. Such first-passage distributions are shown to have exponential-like/geometric-like tails which mimic the behaviour of first-passage distributions in Markov processes even though the holding-time MGFs involved with such semi-Markov processes are typically not rational.

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1. Introduction

Hazard rate functions, and the density/mass and survival functions used in their computation, are fundamental tools used in probability, survival analysis, and reliability. Within the context of the stochastic models commonly used in these fields, such functions can be difficult to compute since the distribution under consideration may only be

specified in terms of its moment generating function (MGF). In such cases, saddlepoint approximations can facilitate the computations, however our aim here is to rather explore asymptotic expansions for all three of these functions. Indeed, a general asymptotic theory for hazard functions has never been formulated in the literature and this is one of our main goals. More generally, the goals of this paper are to formulate an asymptotic theory for all three functions and to develop the theory so it may be applied to the compound distributions and first-passage time distributions commonly dealt with in survival analysis, risk theory, and semi-Markov processes. Our development of such an asymptotic theory relies on using Tauberian theorems, however the standard theorems in Feller (1971, ch. XIII.5) for density and mass functions using "Feller" conditions do not apply to these more complicated compound and first-passage distributions. More inclusive conditions that apply to these distributions are needed in both continuous- and integer-time settings. For the continuous setting, we formulate new "Ikehara" conditions by introducing Tauberian theorems that have been extensively used in analytic number theory, but which have not been previously used (to the authors knowledge) in applied probability. Likewise, in the lattice setting, we introduce very weak Darboux conditions, based on using Darboux's theorem from analytic combinatorics, which apply to compound and first-passage distributions in integer time.

Asymptotic hazard rates are characterized quite generally and are shown to exist under the Ikehara/Darboux conditions needed for tail expansions of density/mass and survival functions. For absolutely continuous distributions, this rate is shown to be $b \geq 0$, the right edge of the convergence region for the associated MGF. For integer-valued distributions, the asymptotic hazard rate is $1 - e^{-b}$.

The most compelling reason for considering Ikehara/Darboux conditions rather than Feller conditions is that they are capable of justifying tail expansions for the infinite mixture/convolution distributions associated with compound distributions, first-passage distributions, and other complicated distributions that occur in stochastic modelling. Among the compound distributions with geometric-like weights, we first consider the ruin distribution in the Cramér-Lundberg and Sparre Andersen models and obtain new expansions for ruin densities and alternative derivations for well-known survival expansions. Ikehara/Darboux conditions also justify expansions for more general compound distributions with negative-binomial-like weights and compound distributions with multivariate weights associated with multiple classes of claim distributions. We show that distributions of the latter type include first-passage distributions in finite-state semi-Markov processes, and this leads to new tail expansions for their density/mass and survival functions which are exponential-like/geometric-like. Thus, first-passage distributions in semi-Markov processes have the same tails as would occur in the more restrictive class of Markov processes and this happens with holding times which are not of phase-type and which do not have rational MGFs. Such exponential/geometric tail expansions reinforce the insensitivity property discussed by Tijms (2003, §5.4) in which semi-Markov

processes mimic the behaviour of Markov processes asymptotically.

The remainder of the paper is organised as follows. Section 2 highlights the main results of the paper and discusses their implications for saddlepoint methods and statistical inference. Section 3 develops expansions for absolutely continuous distributions under Ikehara conditions, and Section 4 considers the analogous results for integer-valued mass functions under Darboux conditions. Section 5 considers finite mixture and convolution applications, and Section 6 discusses compound distributions including the Cramér-Lundberg and Sparre Andersen models. Expansions for first-passage times of semi-Markov processes are in §7. Asymptotic theory when b is a logarithmic singularity is presented in §8.

2. Notation and discussion of main results

Let random variable X have the distribution of interest with MGF $\mathcal{M}(s) = E(e^{sX})$ defined on $\{s \in C : E(e^{sX}) < \infty\}$. Thus, for example, all distributions on $(0, \infty)$ have MGFs which are defined at least on $\{s \in C : \mathrm{Re}(s) \leq 0\}$.

There are four interrelated goals to be achieved in this paper. The first goal is to provide a characterization for the asymptotic hazard rate of X. For absolutely continuous X, Theorem 1 (§3, p. 7) shows that the liminf of the average cumulative hazard rate is $b \in [0, \infty]$, defined as the right edge of the convergence region of the associated MGF. By considering the Cesàro limit rather than the actual hazard limit, this liminf holds without any further conditions on the distribution. If the limiting hazard is known to exist, then b is this limit (Corollary 1). For integer-valued X with hazard sequence $\{h_n\}$, Theorem 2 (§4, p. 13) shows the liminf for the average of $\{-\ln(1 - h_n)\}$ is $b \in [0, \infty]$; thus, a limiting hazard, if it exists, must be $1 - e^{-b}$.

Determining sufficient conditions for the existence of a limiting hazard rate motivates the second goal which is to develop asymptotic expansions for the density and survival function of X to establish such existence. These expansions and the conditions for them depend on the nature of the singularity b for $\mathcal{M}$. When singularity $b > 0$ is gamma-like, so that the MGF is $\mathcal{M}(s) = O(b - s)^{-w}$ as $s \uparrow b$ for $w > 0$, then standard expansions for absolutely continuous X are given by

$$f(t) \sim \frac{g(b)}{\Gamma(w)} t^{w-1} e^{-bt} \qquad \text{and} \qquad S(t) \sim \frac{1}{b} f(t) \qquad t \to \infty \qquad (2.1)$$

where $g(b) = \lim_{s \uparrow b} (b - s)^w \mathcal{M}(s) > 0$. These expansions can be established subject to "Feller" conditions (§3, p. 10) and are justified by using the Hardy-Littlewood-Karamata Tauberian theorem and its extensions from Feller (1971, ch. XIII.5). Such Feller conditions, however, only apply to simple settings and cannot be verified in the more complicated stochastic modelling settings in which X is a compound distribution or a first-passage time for a semi-Markov process. Accommodating these more complicated settings

requires establishing (2.1) under some new more inclusive "Ikehara" conditions which we provide in Proposition 1 (§3, p. 9). Such Ikehara conditions are justified by introducing two Tauberian theorems used exclusively in the field of analytic number theory: the Ikehara-Wiener and Ikehara-Delange theorems, where the former theorem is the main tool for proving the prime number theorem. Thus our two main contributions in developing expansions of the type given in (2.1) are: (i) to replace the restrictive Feller conditions with the new more inclusive Ikehara conditions of Proposition 1, and (ii) to verify that the Ikehara conditions are satisfied for the more complicated compound distributions in stochastic models.

For the setting in which b is a logarithmic singularity, we also propose some Ikehara conditions in Proposition 4 (§8, p. 29) to establish the existence of an asymptotic hazard rate and to justify somewhat different expansions for $f(t)$ and $S(t)$ as $t \to \infty$.

A similar situation occurs when developing mass and survival function expansions for integer-valued X. In the common setting where the MGF has a gamma-like singularity at $b > 0$, so $\mathcal{M}(s) = O(e^b - e^s)^{-w}$ as $s \uparrow b$ for $w > 0$, then a well-known Tauberian theorem from Feller (1971, ch. XIII.5) establishes a Negative Binomial (w, e^{-b})-like tail with expansions

$$p(n) \sim \frac{g(e^b)e^{-bw}}{\Gamma(w)} n^{w-1} e^{-bn} \quad \text{and} \quad S(n) \sim \frac{1}{1-e^{-b}}\, p(n) \qquad n \to \infty \qquad (2.2)$$

where $g(e^b) = \lim_{s \uparrow b}(e^b - e^s)^w \mathcal{M}(s) > 0$. Unfortunately, a condition for using this Tauberian theorem is that $\{p(n)\}$ is monotone in n, and verifying such a condition is difficult when only $\mathcal{M}$ is known. Therefore, the expansions in (2.2) are established under the alternative "Darboux" conditions given in Proposition 2 (§4, p. 13) and afforded by using Darboux's theorem derived from the field of analytic combinatorics. These minimal conditions avoid the monotonicity assumption and apply to the more complicated settings in which X has a compound or first-passage time distribution. Comparable results when $b > 0$ is a logarithmic singularity of $\mathcal{M}$ are given in Proposition 5 (§8, p. 30).

In a large number of practical examples, $b > 0$ is a simple pole so $w = 1$. In such examples, factor $g(b)$ in (2.1) is the negative residue of the MGF at b in the continuous case, while $g(e^b)e^{-b}$ is the negative residue of $\mathcal{M}$ at b in the discrete case. For this simple pole setting, survival and density/mass functions of distributions have exponential-like and geometric-like tails.

These expansions may be broaden to apply to both finite mixture distributions and finite convolutions under either Ikehara/Darboux conditions or Feller conditions. Within this context, our new characterization of the asymptotic hazard rate clarifies an assertion by Block and Savits (2001) that the overall asymptotic hazard rate is the asymptotic hazard rate associated with the strongest and most enduring component within the mixture. This happens because the mixture convergence region is determined by the strongest component having the smallest non-negative convergence region. The same may be said

about convolutions of independent random variables; the strongest addend has MGF whose non-negative convergence region is a proper subset of those for the weaker addends. The main applications for such results include sums and products of independent random variables. For mixture and convolution distributions whose components ostensibly have equal strength and share a common convergence boundary $b > 0$ for their MGFs, we show that the strongest components are those for which the singularity at b attains the highest common order. Applications include sums of i.i.d. random variables.

Our third major goal is to establish these asymptotic expansions in infinite mixture/convolution distributions, such as compound distributions and first-passage distributions in semi-Markov processes, thereby succeeding under Ikehara/Darboux conditions when Feller conditions fail. Examples include density and survival expansions for the ruin amount R in both the Cramér-Lundberg and Sparre Andersen models in Theorem 3 (§6.1, p. 19) and Theorem 4 (§6.2, p. 20). The density expansions are new and have the form $f_R(t) \sim \beta e^{-bt}$, while the survival expansions $S_R(t) \sim \beta e^{-bt}/b$ are well established and have traditionally been proven by using renewal theory as in Feller (1971, XII.5). Once the density expansions have been established, however, the survival expansions follow directly from the smoothing of integration. The converse is not true; the density expansion does not follow from the coarsening effect of differentiating the survival expansion. Thus, the new Ikehara conditions stipulate when both the density and survival functions of R admit exponential expansions. Further examples include general compound distributions with negative-binomial-like weights (Theorem 5 and Corollary 6 in §6.3, p. 21–22), where new density expansions are established to complement the survival expansions of Embrechts et al. (1985) and Willmot (1989). Additional examples include compound distributions determined from multiple classes of claim distributions (Theorems 6 and 7 in §6.3.1, p. 22–24), where new expansions for density and survival functions are established under Ikehara/Darboux conditions.

Our fourth and perhaps most important goal is to extend Cramér-Lundberg-type expansions for density/mass and survival functions so they apply to the broad class of first-passage distributions in general finite-state semi-Markov processes in continuous and integer time. To do this, we first characterise such first-passage distributions as compound distributions determined from multiple classes of claim distributions with multivariate weights as just mentioned; see Proposition 3 (§7, p. 25). This, along with some Ikehara conditions in continuous time, justifies new expansions for first-passage density and survival functions of the form $f(t) \sim \beta e^{-bt}$ and $S(t) \sim \beta e^{-bt}/b$ as given in Theorem 8 (§7, p. 26–27). Here, $b = b(\mathcal{M}) > 0$ denotes the asymptotic failure rate of the first-passage distribution with MGF $\mathcal{M}$ and $\beta = \beta(\mathcal{M}, b)$ is the negative residue of $\mathcal{M}$ at b given explicitly in (7.4). In integer time, first-passage times under minimal Darboux conditions admit geometric-like mass and survival expansions as specified in Theorem 9 (§7, p. 28). Had Feller conditions been used, justification for the $p(n)$ expansion would have required the assumption that $\{p(n)\}$ is monotone in n. The importance of these expansions should

not be understated because the great majority of failure time distributions in applied probability may be formulated as such first-passage times. For example, the ruin distribution in the Cramér-Lundberg and Sparre Andersen models is such a first-passage distribution for the semi-Markov process described in Example 9 (§7, p. 27).

2.1. Implications of results

From Theorems 8 and 9, one may conclude that first-passage time distributions in semi-Markov processes admit the same exponential-like and geometric-like tail expansions that are known to occur for the class of Markov processes. Furthermore, the dominant rate is given by the asymptotic hazard rate b or $1 - e^{-b}$. These findings are the most important results derived by using the general asymptotic theory, and obtaining such results was the original motivation in addressing the whole subject. From the many numerical examples in Butler (2000, 2007 ch. 13), it had already been made clear that first-passage hazard rates approach an asymptote of height b; see the plots of hazard rate functions computed from saddlepoint methods in Butler (2000, 2007 ch. 13). What Theorems 8 and 9 now provide is the theoretical underpinning for the asymptotes in these plots and an explanation for the exponential appearance of the accompanying saddlepoint density and survival plots.

Establishing exponential/geometric tails for such first-passage distributions has important statistical implications for estimating tail probabilities from such distributions using passage-time data. Butler and Bronson (2002, 2012) developed nonparametric bootstrap methods for estimating such probabilities using saddlepoint methods based upon an estimate $\hat{\mathcal{M}}(s)$ for the first-passage MGF. Now, however, rather than estimating $S(t)$ nonparametrically from $\hat{\mathcal{M}}(s)$, expansion estimate $\hat{\beta}e^{-\hat{b}t}/\hat{b}$ can be used instead, where $\hat{b} = b(\hat{\mathcal{M}})$ and $\hat{\beta} = \beta(\hat{\mathcal{M}}, \hat{b})$ are estimates based on $\hat{\mathcal{M}}$. In the context of the Cramér-Lundberg approximation, Chung (2010) has shown in his Ph.D. dissertation that this is indeed better. Starting with the true MGF $\mathcal{M}$, he first showed that expansion $\beta e^{-bt}/b$ is typically more accurate than the Lugananni-Rice saddlepoint approximation for $S(t)$ in the upper quartile of the distribution. Through simulation, he also showed that survival estimate $\hat{\beta}e^{-\hat{b}t}/\hat{b}$ typically has smaller relative error in the upper quartile than a fully nonparametric survival estimate based on $\hat{\mathcal{M}}$ using the methods in Butler and Bronson (2002, 2012).

Another important reason for creating a widely applicable theory for expanding density/mass and survival functions under Ikehara/Darboux conditions is to provide very simple general conditions under which saddlepoint approximations for density/mass and survival functions achieve uniform tail accuracy. Existing conditions in Jensen (1995, §§6.3–6.4) stipulate that distributions must satisfy relatively complicated conditions related to his method of proof which can be difficult to verify. The much simpler Ikehara conditions of Proposition 1 suffice when tails are gamma-like, and the author has recently

shown (in new unpublished work) that saddlepoint approximations for density and survival functions achieve limiting relative error given by Stirling's approximation for $\Gamma(w)$. For lattice distributions, the same results hold for saddlepoint approximations of mass and survival functions under minimal Darboux conditions. Insofar as Ikehara/Darboux conditions ensure that such expansions apply to compound distributions and first-passage distributions in semi-Markov processes, then such uniform tail accuracy also carries over when saddlepoint methods are used to approximate such distributions. Thus this work generalises and simplifies the uniformity results derived in Jensen (1995, ch. 7) for compound distributions and extends the uniformity results to first-passage distributions.

3. Absolutely continuous distributions

Suppose X is an absolutely continuous random variable with support $(0, \infty)$, density $f(t)$, and survival function $S(t) = 1 - F(t)$. The hazard and cumulative hazard rate functions are

$$h(t) = f(t)/S(t) \qquad \text{and} \qquad H(t) = \int_0^t h(x)dx = -\ln\{S(t)\}.$$

The associated MGF is defined as $\mathcal{M}(s) = E(e^{sX})$ and converges on the real line for either $s \in (-\infty, b)$ or $(-\infty, b]$ for $b \geq 0$. The limiting average hazard rate is now characterised in terms of its MGF.

Theorem 1. *If a non-negative absolutely continuous random variable X has moment generating function $\mathcal{M}(s)$ converging on $(-\infty, b)$ or $(-\infty, b]$ for $b \geq 0$, then*

$$\liminf_{t \to \infty} \frac{H(t)}{t} = b.$$

The theorem can be derived from first principles (see Supplementary Materials, §A.1.1) or by using Theorem 2.4e from Widder (1946, p. 44).

Lemma 1. *(Widder). Suppose Laplace-Stieltjes transform $\mathcal{G}(s) = \int_0^\infty e^{-st}dG(t)$ converges on $\mathrm{Re}(s) > -b < 0$ for some function $G(t)$ of bounded variation. Then, $G(\infty)$ exists and*

$$-b = \limsup_{t \to \infty}\{t^{-1}\ln|G(\infty) - G(t)|\}.$$

Proof of Theorem 1. Let $G(t) = F(t)$ so that $G(\infty) = 1$ and

$$b = -\limsup_{t \to \infty}\{t^{-1}\ln S(t)\} = \liminf_{t \to \infty}\{-t^{-1}\ln S(t)\} = \liminf_{t \to \infty}\{t^{-1}H(t)\}.$$

Theorem 1 generalises to apply to any absolutely continuous random variable X with distribution on $(-\infty, \infty)$. If X has a MGF which converges on (a, b) or $(a, b]$, where

$a \leq 0 \leq b$, then $\liminf_{t \to \infty} H(t)/t = b$ as shown in §A.1.2. For example, the Cauchy distribution has $a = 0 = b$ and $\lim_{t \to \infty} H(t)/t = 0$.

If the limiting hazard rate exists, as it does for many commonly used distributions, then these liminfs are indeed limits.

Corollary 1. *For an absolutely continuous random variable X with support on $(-\infty, \infty)$, if* $\lim_{t \to \infty} h(t)$ *exists, then*

$$\lim_{t \to \infty} h(t) = \lim_{t \to \infty} \frac{H(t)}{t} = b. \tag{3.1}$$

Proof. If $h(t) \to b_0$ then the Cesàro mean $t^{-1} H(t) = t^{-1} \int_0^t h(s)ds \to b_0$ as $t \to \infty$. A proof of this follows the same approach as used for sequences. By Theorem 1, this limit must be b so $b_0 = b$ and (3.1) holds. $\square$

While this is the characterization we seek, the presumption that $h(t)$ has a limit is a fact that would not typically be known for a new unfamiliar distribution. Thus, the benefit of the corollary is to eliminate the computation but only if the limit is known to exist. Sufficient conditions are needed to guarantee such a limit and are provided below. The following pathological example provides some guidance for determining what these sufficient conditions need to be. The distribution has a periodic hazard rate function with no limit and has $\liminf_{t \to \infty} h(t)$ different from $\liminf_{t \to \infty} H(t)/t$.

Example 1. The density

$$f(t) = 2/3 \left(1 + \sin t\right) e^{-t} \qquad t > 0 \tag{3.2}$$

takes value 0 on the set $\{3\pi/2 + 2\pi k : k = 0, 1, \ldots\}$ so that $\liminf_{t \to \infty} h(t) = 0$. The hazard rate is

$$h(t) = \frac{2(1 + \sin t)}{2 + \cos t + \sin t},$$

a 2π-periodic function that does not have a limit. Its MGF is

$$\mathcal{M}(s) = 2/3 \left[(1 - s)^{-1} + \{(1 - s)^2 + 1\}^{-1} \right] \qquad \mathrm{Re}(s) < 1 = b. \tag{3.3}$$

Direct computation shows that

$$t^{-1} H(t) = 1 + t^{-1} \left(\ln 3 - \ln[2 \cos^2(t/2) + \{\cos(t/2) + \sin(t/2)\}^2] \right).$$

The coefficient of t^{-1} is bounded so that $\lim_{t \to \infty} H(t)/t = 1$.

The lack of a limiting hazard rate in this example can be explained by $\mathcal{M}$ not having a dominant pole on the boundary $\{s \in C : \mathrm{Re}(s) = 1\}$ of its convergence region. From

(3.3), we see that it has three simple poles $\{1, 1 \pm i\}$, which all vie for dominance of the hazard rate, and this leads to the periodic behaviour of $h(t)$. To ensure the existence of a limiting hazard function, we exclude such MGFs by stipulating some new Ikehara conditions $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{UND}$ in the next result.

Proposition 1. *Let X have an absolutely continuous distribution $F(t)$ with support $(0, \infty)$ and associated moment generating function $\mathcal{M}(s)$ that converges on the complex half-plane $\{s \in C : \mathrm{Re}(s) < b\}$ for $b > 0$. Let X also satisfy the Ikehara conditions as given below. Then $\lim_{t \to \infty} h(t) = b$,*

$$f(t) \sim \frac{g(b)}{\Gamma(w)} t^{w-1} e^{-bt} \qquad \text{and} \qquad S(t) \sim \frac{1}{b} f(t) \tag{3.4}$$

as $t \to \infty$, where $g(b) = \lim_{s \uparrow b} (b - s)^w \mathcal{M}(s)$.

Ikehara conditions: *X (or F or $\mathcal{M}$) satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{UND}$ where*
($\mathfrak{I}_{\mathcal{M}}$) b is a dominant singularity in that the analytic continuation of $\mathcal{M}$ may be expressed as

$$\mathcal{M}(s) = g(s)(b - s)^{-w} + h(s), \tag{3.5}$$

where $w > 0$, g and h are analytic on $\{s \in C : \mathrm{Re}(s) \leq b\}$, and $g(b) \neq 0$; and
($\mathfrak{I}_{UND}$) There exists an $\varepsilon > 0$ such that the $(b + \varepsilon)$-tilted improper density $f_{b+\varepsilon}(t) := \exp\{(b + \varepsilon)t\} f(t)$ is non-decreasing for $t > A$, for some A.

If w is not a positive integer, then Ikehara condition $\mathfrak{I}_{\mathcal{M}}$ has the multi-function factor $(b - s)^{-w}$ which assumes principal branch values that are real-valued for $s < b$ and makes use of a branch cut along $[b, \infty]$.

These results state that gamma-like MGFs have densities with gamma-like tails. While such conclusions are not new, the Ikehara conditions $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{UND}$ for making such conclusions are new to the field of probability. A proof of Proposition 1 is given in §B.1.3 and follows from two Tauberian theorems that have mostly been used in analytic number theory. In the case of a simple pole ($w = 1$) at b, this includes the Ikehara-Wiener theorem, given in Theorem B1 of §B.1.1, which is well-known as the primary tool for proving the prime number theorem; see Widder (1946, pp. 233–236) for its use in the proof. Other versions of this theorem are also described in, for example, Chandrasekharan (1968, p. 124), Doetsch (1950, p. 524), or Korevaar (2004, thm. 4.2, p. 124). For other cases in which $0 < w \neq 1$, the proof uses the lesser known Ikehara-Delange theorem as stated in Theorem B2 of §B.1.2 and given in Narkiewiez (1983, thm. 3.9, p. 119).

Proposition 1 also holds if Ikehara conditions are replaced with the following Feller conditions, which are those needed to use results based on the Hardy-Littlewood-Karamata theorem in Feller (1971, §XIII.5); see §A.2.1 for a proof.

Feller conditions: X *satisfies* $\mathfrak{F}_{\mathcal{M}} \cap \mathfrak{F}_{UM}$ *where*

($\mathfrak{F}_{\mathcal{M}}$) *For real* s, $M(s) \sim g(s)(b-s)^{-w}$ *as* $s \uparrow b$ *for* $w > 0$, *and* g *is left-continuous at* b *with* $g(b) > 0$; *and*

($\mathfrak{F}_{UM}$) *The improper b-tilted density* $f_b(t) = e^{bt}f(t)$ *is ultimately monotone, i.e. it is monotone for all* $t > A$, *for some* A.

For Example 1, note that Feller condition $\mathfrak{F}_{UM}$ fails to hold since tilted density $e^t f(t) = 2/3\,(1 + \sin t)$ is not ultimately monotone in t as $t \to \infty$. Overall, condition $\mathfrak{F}_{UM}$ can be weakened to the condition $\mathfrak{F}_{UM2}$ that $f_b(t) \sim v(t)$ as $t \to \infty$ with $v(t)$ ultimately monotone as indicated in §A.2.1.

In many simple practical applications, both the Feller and Ikehara conditions apply. For example, if $X = -\ln\{\text{Beta}\,(\alpha, \beta)\}$, then both Ikehara and Feller conditions hold for all values of $\alpha, \beta > 0$; this gives asymptotic hazard rate α and tail behaviour $f(t) \sim \Gamma(\alpha + \beta)/\{\Gamma(\alpha)\Gamma(\beta)\}e^{-\alpha t}$.

In more complicated stochastic model settings, however, this is not the case and only the Ikehara conditions can be applied in this broader range of settings. Direct comparison of the two sets of conditions shows why the Ikehara conditions are more practically useful. Verifying the condition placed on density f is the main difficulty. Feller condition $\mathfrak{F}_{UM}$ supposes the b-tilted density is either ultimately non-decreasing or non-increasing with almost all applications being ultimately non-decreasing. Ikehara condition $\mathfrak{I}_{UND}$ supposes there exists some $(b + \varepsilon)$-tilted density, with $\varepsilon > 0$, that is ultimately non-decreasing. Feller condition $\mathfrak{F}_{UM}$ (applied as ultimately non-decreasing) is much stronger and more restrictive and implies that Ikehara condition $\mathfrak{I}_{UND}$ holds for all $\varepsilon > 0$.

The main consequence of using the more restrictive Feller condition $\mathfrak{F}_{UM}$ is that it is generally not possible to show that it holds in stochastic modelling settings whereas the more relaxed Ikehara condition $\mathfrak{I}_{UND}$ is often easily shown to hold for a sufficiently large $\varepsilon > 0$. The classic Cramér-Lundberg example of §6 provides an example. In this model, the ruin amount R with density $f_R(t)$ has MGF of the form

$$\mathcal{M}_R(s) = \frac{1 - \rho}{1 - \rho\mathcal{M}_E(s)} \qquad \text{Re}(s) < b. \qquad (3.6)$$

Here, $\mathcal{M}_E$ is the MGF for the excess life distribution of the claim density $f_X(t)$ and convergent on $\{\text{Re}(s) < c\}$, while $b \in (0, c)$ is the smallest positive zero of the denominator in (3.6). We want to conclude that the associated ruin density $f_R(t) \sim c_1 e^{-bt}$ as $t \to \infty$ for constant $c_1 > 0$ as stated in Theorem 3 of §6.1. Assuming the claim density f_X satisfies Feller condition $\mathfrak{F}_{UM}$ does not allow one to conclude that ruin density f_R also satisfies $\mathfrak{F}_{UM}$ since under the former assumption $e^{ct}f_X(t)$ is ultimately non-decreasing whilst under the latter assumption $e^{bt}f_R(t)$ must be ultimately non-decreasing. The problem is simply that $b < c$. Such problems are avoided by placing an Ikehara condition on f_X. As will be seen in §6, a uniform Ikehara assumption on f_X, in which an $\varepsilon > 0$ exists for

which $e^{(c+\varepsilon)t}f_X(t)$ is non-decreasing for all $t > 0$, allows one to conclude the same uniform Ikehara property for f_R, i.e. $e^{(c+\varepsilon)t}f_R(t)$ is also non-decreasing for all $t > 0$. Thus, for this and other stochastic models, an Ikehara condition needed for Proposition 1 to apply to the intractable density f_R can be deduced by assuming the same Ikehara condition on the more tractable input claim density f_X. This idea underlies all the asymptotic results developed in the major applications concerning compound and first-passage densities in §§6–7.

Further comparison of Ikehara and Feller conditions placed upon $\mathcal{M}$ reveals that the Ikehara condition $\mathfrak{I}_\mathcal{M}$ is stronger than the corresponding Feller condition $\mathfrak{F}_\mathcal{M}$ thus compensating for the weaker condition placed on f. However, in most all practical settings, both conditions $\mathfrak{I}_\mathcal{M}$ and $\mathfrak{F}_\mathcal{M}$ tend to hold together and showing either is typically quite straightforward when $\mathcal{M}$ is given.

Example 2. (Excess life distribution). Suppose absolutely continuous X satisfies all the conditions of Proposition 1. If X is interpreted as an interarrival time, then the excess life E associated with it has density $f_E(t) = S(t)/\mu$, with $\mu = E(X)$, and MGF $\mathcal{M}_E(s) = \{1 - \mathcal{M}(s)\}/(-\mu s)$ which is also convergent on $\{\operatorname{Re}(s) < b\}$. If X satisfies the Ikehara conditions $\mathfrak{I}_\mathcal{M} \cap \mathfrak{I}_{UND}$ then so does E if the singularity b for $\mathcal{M}$ is restricted to being a w-pole (so $w > 0$ is an integer); see §B.2.1 for a proof. A comparable result can be shown under Feller conditions; see §A.2.2.

From a measure-theoretic point of view, Proposition 1 applies only to a Radon-Nykodym derivative f that satisfies either $\mathfrak{I}_{UND}$ or $\mathfrak{F}_{UM}$. In most applications there is no ambiguity since f is ultimately continuous with at most a finite number of step discontinuities. The theorem can also allow f to have an infinite number $\{t_n : n \geq 1\}$ of step discontinuities that extend into the tail. Under such conditions, both $\mathfrak{I}_{UND}$ and $\mathfrak{F}_{UM}$ may hold if all but a finite number are upward stepping so that ultimately $f(t_n^-) \leq f(t_n^+)$; if, however, $f(t_n^-) > f(t_n^+)$ i.o., then neither of the conditions can hold.

Proposition 1 may be extended to absolutely continuous distributions supported on the real line using slightly amended Ikehara conditions and Feller conditions; see Corollaries B1 and A1 respectively in §§B.2.2 and A.2.3. As an example, consider $X = -\ln\{\operatorname{Gamma}(\alpha,\beta)\}$ with MGF $\mathcal{M}(s) = \beta^s \Gamma(\alpha - s)/\Gamma(\alpha)$ which has a dominant singularity at α. Both Corollaries B1 and A1 apply to give tail behaviour $f(t) \sim \beta^\alpha e^{-\alpha t}/\Gamma(\alpha)$ which is easily verified directly.

While the expansions in (3.4) of Proposition 1 apply to gamma-like distributions, they do not apply to heavy-tailed distributions on $(0, \infty)$, whose MGFs converge on the non-open region $(-\infty, 0]$. Existing methods for obtaining such expansions with subexponential distributions do not lead to tail approximations with the same accuracy and hence practical importance as the current light-tailed expansions in Proposition 1; see Tijms (2004, p. 332–333) and Rolski et al. (1999, §5.4.2) for discussion and numerical

verification. Neither does Proposition 1 apply to very light-tailed distributions, such as a Normal (μ, σ^2), for which $b = \infty$ and whose MGF lacks finite singularities. Thirdly, it does not deal with all distributions for which b is a branch point of the MGF; e.g. an inverse Gaussian MGF, which converges on non-open region $(-\infty, b]$, as well as other examples given in §A.4. Expansions for such distributions are considered in related unpublished work by the author. Finally it does not deal with branch points created from the logarithm multi-function; such examples are covered in §8. Even though the theorem does not apply to such distributions, the value for $\liminf_{t \to \infty} H(t)/t$ is still $b \in [0, \infty]$, as described in Theorem 1, and this conclusion does not depend upon the type of singularity at b nor upon whether $b = 0$ or ∞.

3.1. Large deviation theory and numerical accuracy of the expansions

Large deviation theory is concerned with the decay of $S(t)$ as $t \to \infty$ and a typical theorem would show that $-b < 0$ is the exponential rate of decay for S as expressed through the equality $\lim_{t \to \infty} t^{-1} \ln S(t) = -b$. The conclusions of Proposition 1, however, are stronger because they not only imply such results but also provide the rate of such convergence as expressed through the leading term in the expansion of $t^{-1} \ln S(t) + b = o(1)$. Consider, for example, the very common setting in which b is a simple pole. Then, Proposition 1 gives

$$t^{-1} \ln S(t) + b = t^{-1} \ln\{g(b)/b\} + o(t^{-1}) \qquad t \to \infty, \tag{3.7}$$

so the leading term in $o(1)$ is $t^{-1} \ln\{g(b)/b\}$ to order $o(t^{-1})$.

The use of the approximation in (3.7), as opposed to $t^{-1} \ln S(t) \approx -b$, is particularly important when approximating tail probabilities in practical applications. Indeed, expression (3.7) can be quite accurate when b is a dominant pole even for moderately large values of t. Alternatively, $t^{-1} \ln S(t) \approx -b$ is only accurate for extremely large values of t (Barndorff-Nielsen and Cox, 1989, §6.6). As an example, Tijms (2003, §8.4, Table 8.4.1) shows the good numerical accuracy that can be obtained when using (3.7) in the context of the well-known Cramér-Lundberg approximation which is to be discussed in §6.1.

4. Lattice distributions

It suffices to consider a non-negative integer-valued random variable with mass function $\{p(n) : n \geq 0\}$ and hazard rate sequence

$$h_n = \frac{p(n)}{\sum_{j=n}^{\infty} p(j)} \qquad n \geq 0.$$

Theorem 2. *If X has a mass function on the non-negative integer lattice and moment generating function $\mathcal{M}(s)$ which converges on $(-\infty, b)$ or $(-\infty, b]$, for $b \geq 0$, then*

$$\liminf_{n \to \infty} \left\{ -\frac{1}{n} \sum_{k=0}^{n-1} \ln(1 - h_k) \right\} = b. \tag{4.1}$$

Proof. The proof is either derived from first principles (see §A.1), or directly from Lemma 1 by noting that

$$\ln S(n) = \ln P(X \geq n) = \ln \prod_{k=0}^{n-1} (1 - h_k) = \sum_{k=0}^{n-1} \ln(1 - h_k).$$

The result easily extends to arbitrary distributions on the integer lattice in which the MGF converges on (a, b) or $(a, b]$ for $a \leq 0 \leq b$. Following the proof in the continuous case, then $\liminf_{n \to \infty} \{ -n^{-1} \sum_{k=-\infty}^{n-1} \ln(1 - h_k) \} = b$.

Sufficient conditions for the existence of a limiting hazard rate as well as asymptotic expansions for the mass function $p(n)$ and survival function $S(n)$ are now given.

Proposition 2. *Suppose X has non-negative integer support, and its moment generating function $\mathcal{M}(s)$, which converges on $\{s \in C : \mathrm{Re}(s) < b\}$ with $b > 0$, satisfies Darboux condition $\mathfrak{D}_{\mathcal{M}}$ below. Then, $\lim_{n \to \infty} h_n = 1 - e^{-b}$,*

$$p(n) \sim \frac{g(e^b) e^{-bw}}{\Gamma(w)} n^{w-1} e^{-bn}, \qquad \text{and} \qquad S(n) \sim \frac{1}{1 - e^{-b}} p(n) \tag{4.2}$$

as $n \to \infty$.

($\mathfrak{D}_{\mathcal{M}}$) $\mathcal{M}$ has the form

$$\mathcal{M}(s) = g(e^s)(e^b - e^s)^{-w} + h(e^s), \tag{4.3}$$

where $w > 0$, $g(e^s)$ and $h(e^s)$ are analytic on $\{s \in C : \mathrm{Re}(s) \leq b\}$ with $g(e^b) \neq 0$, and $(e^b - e^s)^{-w}$ assumes principal branch values for non-integer w. (Condition $\mathfrak{D}_{\mathcal{M}}$ ensures that b is a dominant singularity, i.e. it is the only singularity on the boundary of the principal convergence region defined as $\{s \in C : -\pi < \mathrm{Im}(s) \leq \pi; \mathrm{Re}(s) = b\}$.)

The novelty and importance of Proposition 2 are the Darboux condition $\mathfrak{D}_{\mathcal{M}}$ which ensures the well-known results in (4.2). Established conditions for deriving (4.2) from Tauberian Theorem 5 of Feller (1971, XIII.5) require the additional assumption that $\{p(n)\}$ is ultimately non-increasing. Such an assumption is difficult to verify from $\mathcal{M}$ alone and this undermines all our applications to stochastic modelling. The Darboux condition $\mathfrak{D}_{\mathcal{M}}$, by comparison, does not limit such applications as it is quite weak and can be easily verified from $\mathcal{M}$.

A proof of Proposition 2 is given in §A.5.2 and uses Darboux's theorem, which may be considered a lattice version of the Ikehara-Delange and Ikehara-Wiener theorems. Theorem A1 of §A.5.1 is a modification of Darboux's theorem, as given in Theorem 5.11 of Wilf (2006, p. 194), and deals with generating functions (GFs) that converge on a disc of radius $c > 0$ rather than the usual radius of $c = 1$ in which the theorem is generally stated. The proof also uses the Stolz–Cesàro theorem in Lemma A1 of §A.5.1 which is a discrete version of l'Hôpital's rule; see Huang (1988, p. 322).

For a simple distribution such as $X \sim$ Negative Binomial (m, p), Proposition 2 applies with $w = m$ and $e^{-b} = p$ and the asymptotic order in (4.2) is easily verified. The expressions in (4.2) are exact for the Geometric mass function with $m = 1$. Our next class of examples is less trivial.

Example 3. (Equilibrium distributions). Suppose $\{p(n) : n \geq 0\}$ has GF $\mathcal{P}(z)$ and is the equilibrium distribution for a positive recurrent queue with a countably infinite state space. Conditions on $\mathcal{P}$ that ensure $p(n) \sim \gamma e^{-bn}$ as $n \to \infty$ are given in Theorem C.1 of Tijms (2003, pp. 452–453) and they agree with the conditions in Proposition 2. The GF has form $\mathcal{P}(z) = \mathcal{N}(z)/\mathcal{D}(z)$ and is assumed to be analytic on $\{|z| \leq c\}$ apart from a simple pole at $c = e^b > 1$, which results from a simple zero of $\mathcal{D}(z)$. When written as in (4.3) of Proposition 2, the MGF $\mathcal{P}(e^s) = g(e^s)/(e^b - e^s)$ has factor $g(e^s) = \mathcal{N}(e^s)(e^b - e^s)/\mathcal{D}(e^s)$, which is analytic on $\{\mathrm{Re}(s) \leq b\}$, and $\mathcal{P}(e^s)$ admits a negative residue at b given by

$$\gamma = g(e^b)e^{-b} = -\mathcal{N}(e^b)/\{e^b \mathcal{D}'(e^b)\}.$$

A wide range of examples in Tijms (2003) result in equilibrium distributions with GFs of this form. Examples includes a discrete-time queue (Example 3.4.1), a continuous-time Markov process on a semi-infinite state space (§4.4), the M/G/1 queue (§§4.4 and 9.2), a bulk M^X/G/1 queue (§9.3), and approximations to several GI/G/m queues (§§9.5–9.7).

Proposition 2 can be extended to lattice distributions over all integers. The results are summarised in Corollary A6 in §A.5.3 under slightly modified conditions.

5. Finite convolution and mixture distributions

A considerable broadening of the theory in §§3 and 4 results when it is applied to finite convolutions and mixtures. Let X be an absolutely continuous or integer-valued random variable that has an asymptotic hazard by virtue of satisfying the Ikehara or Darboux conditions of Proposition 1 or 2. Now, convolve X with independent variable Y and mix the resulting distribution with the distribution of independent random variable Z. This leads to a mixture-convolution random variable W with density/mass function

$pf_{X+Y}(t) + qf_Z(t)$ for some $p \in (0,1]$ with $q = 1 - p$. Conditions are given below to ensure that the distribution of W also satisfies Proposition 1 or 2 so that its asymptotic hazard rate exists and its tail behaviour can be characterised.

5.1. One strongest variable

Let Y and Z be strictly weaker components than X in the sense that their MGFs have strictly larger non-negative convergence regions than that for X. Then, apart from some additional technical conditions, the mixture-convolution distribution for W has the same asymptotic hazard rate as that for the strongest variable X. Also, the survival and density/mass functions for W have the same gamma-like and negative binomial-like tails as for X and differ only by having a different value for constant $g(b)$. The general interpretation that may be given is that the strongest component prevails asymptotically, and the two weaker components Y and Z only express themselves by changing the value of the constant $g(b)$. Block *et al.* (1993) and Block and Joe (1997, thm. 4.1) pointed out the lack of influence of Z on asymptotic hazard whilst the minimal influence of Y is new. These results are formalized in Corollary 2 under Ikehara conditions and the proof is relegated to §B.3.1. Comparable results under Feller conditions are given in Corollary A2 of §A.3.1. We use subscripted notation so $\mathcal{M}_X(s)$ denotes the MGF of X, etc.

Corollary 2. *Let X, Y, and Z be absolutely continuous and non-negative variables such that X is stronger than Y and Z; i.e. let $\mathcal{M}_X(s), \mathcal{M}_Y(s)$, and $\mathcal{M}_Z(s)$ converge on $\{\mathrm{Re}(s) < b\}, \{\mathrm{Re}(s) < b + \eta_Y\}$, and $\{\mathrm{Re}(s) < b + \eta_Z\}$ respectively for $b > 0$ and some values $\eta_Y > 0 < \eta_Z$.*

Ikehara conditions: *Suppose X satisfies $\mathfrak{I}_{\mathcal{M}}$ and either X or Y (or both) satisfies the uniform Ikehara condition denoted by $\mathfrak{I}_{ND(0,\infty)}$ and given below. Let Z satisfy condition $\mathfrak{I}_{UND}$ if $p < 1$.*

$(\mathfrak{I}_{ND(0,\infty)})$ X satisfies $\mathfrak{I}_{ND(0,\infty)}$ if an $\varepsilon > 0$ exists for which $e^{(b+\varepsilon)t}f_X(t)$ is non-decreasing for all $t > 0$.

Then, $X + Y$ satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{ND(0,\infty)}$ and W satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{UND}$. Thus, W has asymptotic hazard rate $b > 0$, density

$$f_W(t) \sim p\mathcal{M}_Y(b)f_X(t) \sim p\mathcal{M}_Y(b)g_X(b)\Gamma(w)^{-1}t^{w-1}e^{-bt} \qquad t \to \infty, \qquad (5.1)$$

and survival $S_W(t) \sim f_W(t)/b$.

Example 4. (Sums and products of independent variables). Suppose $Z_1, \ldots, Z_k$ are independent with $Z_i \sim \mathrm{Gamma}\,(\alpha_i, \beta_i)$ and $\beta_1 < \min_{i \geq 2} \beta_i$. The sum $W = Z_1 + \sum_{i \geq 2} Z_i$ is the passage time though a series connection of states in a semi-Markov process with

Gamma (α_i, β_i) holding times. The strongest addend of X is Z_1. Ikehara conditions require $\min_i \alpha_i \geq 1$, since the conditions of Corollary 2 are that at least one Z_i satisfies $\mathfrak{I}_{ND(0,\infty)}$. The Feller conditions of Corollary A2 in §A.3.1 are more restrictive and require $\alpha_1 \geq 1$ since they are more specific in requiring that the strongest variable satisfy the uniform Feller condition $\mathfrak{F}_{ND(0,\infty)}$. Both sets of conditions lead to the conclusion that W has a Gamma (α_1, β_1) tail.

If $Z_1, \ldots, Z_k$ are independent with $Z_i \sim$ Beta (α_i, β_i) and $\alpha_1 < \min_{i \geq 2} \alpha_i$, then the product $\prod_{i \geq 1} Z_i$ arises as the posterior distribution for the probability that a series connection of k independent components is working. It is also the null distribution for Wilks' likelihood ratio test and most of the other likelihood ratio test statistics in multivariate analysis of variance; see Anderson (2003, ch. 9–10). Variable $W = -\ln Z_1 - \sum_{i \geq 2} \ln Z_i$ has strongest addend $-\ln Z_1$ and Corollaries 2 and A2 imply that W has an exponential-like tail of order $O(e^{-\alpha_1 t})$ under Ikehara condition $\max_i \beta_i \geq 1$ or Feller condition $\beta_1 \geq 1$.

For absolutely continuous densities, the additional uniformity restriction of $\mathfrak{I}_{ND(0,\infty)}$ in Corollary 2 over $\mathfrak{I}_{UND}$ does not adversely restrict the range of applicability of the corollary. Indeed, the only practical densities in $\mathfrak{I}_{UND} \backslash \mathfrak{I}_{ND(0,\infty)}$ seem to be those that are either unbounded at $t = 0$ or have a downward stepping discontinuity, i.e. $f(t^-) > f(t^+)$ for some $t > 0$. Also, only one of the two variables X and Y needs to satisfy $\mathfrak{I}_{ND(0,\infty)}$ while the other may have an unbounded density or may have downward stepping discontinuities.

When the random variables in Corollary 2 have support on $(-\infty, \infty)$, then comparable results can be derived and are given in Corollary B2 of §B.3.2. The same results using Feller conditions are given in Corollary A3 of §A.3.2. As an example, consider the gamma variables in Example 4 and suppose $\alpha_1 < \min_{i \geq 2} \alpha_i$. The product $\prod_{i \geq 1} Z_i$ describes the distribution for the determinant of a $k \times k$ Wishart matrix based on independent components. Taking logarithms, then $W = -\ln Z_1 - \sum_{i \geq 2} \ln Z_i$ has strongest addend $-\ln Z_1$ with an exponential-like tail of order $O(e^{-\alpha_1 t})$ under both Ikehara and Feller conditions.

Corollary 3. *Suppose integer-valued $X \geq 0$ satisfies the conditions of Proposition 2 with $\mathcal{M}_X(s) = g_X(e^s)(e^b - e^s)^{-w} + h_X(e^s)$. Let independent integer-valued variables $Y \geq 0$ and $Z \geq 0$ be such that $\mathcal{M}_Y(s)$ and $\mathcal{M}_Z(s)$ are analytic on $\{s \in C : \operatorname{Re}(s) < b + \eta_Y\}$ and $\{s \in C : \operatorname{Re}(s) \leq b + \eta_Z\}$ respectively for $\eta_Y > 0 < \eta_Z$. Then, the distribution of W has asymptotic hazard rate $1 - e^{-b}$ and mass and survival functions of asymptotic order given in (4.2) with $g(e^b)$ replaced by $p\mathcal{M}_Y(b)g_X(e^b)$.*

5.2. Equally strong variables

Suppose random variables X, Y, and Z are ostensibly of equal strength with MGFs that share the common convergence region $\{s \in C : \operatorname{Re}(s) < b\}$. Subject to some Ikehara con-

ditions, convolution/mixture variable W has asymptotic hazard rate b with asymptotic tail behaviour as given below. See §B.3.3 for a proof. Comparable results under Feller conditions are given in Corollary A4 in §A.3.3.

Corollary 4. *Suppose absolutely continuous, non-negative, and independent variables X, Y, and Z have moment generating functions $\mathcal{M}_X(s), \mathcal{M}_Y(s)$, and $\mathcal{M}_Z(s)$ which share the common convergence region $\{s \in C : \mathrm{Re}(s) < b\}$.*

Ikehara conditions: *Let X, Y, and Z all satisfy $\mathfrak{I}_\mathcal{M}$ of Proposition 1, and suppose the singularities for $\mathcal{M}_X(s), \mathcal{M}_Y(s)$, and $\mathcal{M}_Z(s)$ at $b > 0$ are poles with positive integer orders w_X, w_Y, and w_Z respectively. Furthermore, suppose either X or Y satisfies $\mathfrak{I}_{ND(0,\infty)}$ and Z satisfies $\mathfrak{I}_{UND}$ if $p < 1$.*

Then, $X + Y$ satisfies $\mathfrak{I}_\mathcal{M} \cap \mathfrak{I}_{ND(0,\infty)}$ and W satisfies $\mathfrak{I}_\mathcal{M} \cap \mathfrak{I}_{UND}$. Thus, W has asymptotic hazard rate $b > 0$, density

$$f_W(t) \sim g_W(b)\Gamma(w_*)^{-1}t^{w_*-1}e^{-bt} \qquad t \to \infty, \tag{5.2}$$

and survival $S_W(t) \sim f_W(t)/b$, where $w_* = \max\{w_X + w_Y, w_Z\}$ and

$$g_W(b) = \begin{cases} pg_X(b)g_Y(b) & \text{if} \quad w_X + w_Y > w_Z \\ pg_X(b)g_Y(b) + qg_Z(b) & \text{if} \quad w_X + w_Y = w_Z \\ qg_Z(b) & \text{if} \quad w_X + w_Y < w_Z \end{cases}. \tag{5.3}$$

When variables in a finite convolution/mixture share a common convergence region, then, according to Corollary 4, the resulting distribution still reflects the strongest component but that component is now the one with the highest order singularity at b. If multiple components share the highest common order, as occurs when $w_X + w_Y = w_Z$ in (5.3), then all such components contribute to the asymptotic order through the value of coefficient $g_W(b)$. The following result for i.i.d. variables follows directly from Corollary 4. The same result under Feller conditions is given in Corollary A5 of §A.3.4.

Corollary 5. *(Convolution of i.i.d. variables). Let $W = X_1 + \cdots + X_m$ where $X_1, \ldots, X_m$ are non-negatively-valued i.i.d. variables from an absolutely continuous distribution.*

Ikehara conditions: *Suppose X_1 satisfies $\mathfrak{I}_\mathcal{M} \cap \mathfrak{I}_{ND(0,\infty)}$ and singularity $b > 0$ for $\mathcal{M}_{X_1}$ is a w-pole, for integer w.*

Then, W also satisfies $\mathfrak{I}_\mathcal{M} \cap \mathfrak{I}_{ND(0,\infty)}$. Proposition 1 applies to give asymptotic hazard rate $b > 0$, density

$$f_W(t) \sim g_{X_1}^m(b)\Gamma(mw)^{-1}t^{mw-1}e^{-bt} \qquad t \to \infty, \tag{5.4}$$

and survival function $S_W(t) \sim f_W(t)/b$. If $\mathcal{M}_{X_1}(s)$ takes the form $g_{X_1}(s)(b-s)^{-w}$ in (3.5) with addend $h_{X_1}(s) \equiv 0$, then the same conclusions hold for arbitrary $w > 0$ (it need not be an integer).

Proof. We only comment on showing that W satisfies $\mathfrak{I}_{ND(0,\infty)}$. Apply Lemma B1 of §B.3.1 successively to the sequence $W_2 = X_1 + X_2$, $W_3 = W_2 + X_3, \ldots, W = W_{m-1} + X_m$ to show that $\{W_2, \ldots, W_{m-1}, W\}$ all satisfy condition $\mathfrak{I}_{ND(0,\infty)}$. Thus W satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{ND(0,\infty)}$. $\square$

Example 5. (Excess life distribution). Consider an m-fold convolution $W = E_1 + \cdots + E_m$ of i.i.d. excess life variables E as in Example 2. Two results are shown in §B.3.4. First, if interarrival time X satisfies $\mathfrak{I}_{\mathcal{M}}$ with $\mathcal{M}_X$ convergent on $\{\mathrm{Re}(s) < b\}$ and $b > 0$ is an w-pole, then E satisfies $\mathfrak{I}_{\mathcal{M}}$ and $b > 0$ is a w-pole for $\mathcal{M}_E$.

Secondly, if X satisfies $\mathfrak{I}_{ND(0,\infty)}$, then E satisfies $\mathfrak{I}_{ND(0,\infty)}$. Putting the two results together, if X satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{ND(0,\infty)}$ with b as an w-pole of $\mathcal{M}_X$, then E satisfies the requirements for Corollary 5, i.e. E satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{ND(0,\infty)}$ and singularity b is a w-pole for $\mathcal{M}_E$. Thus, W also satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{ND(0,\infty)}$ and has density and survival function with asymptotic orders as in (5.4).

6. Compound Distributions

A rich Tauberian theory is derived below for such infinite mixture distributions under Ikehara conditions. Corresponding results cannot be derived under Feller conditions since it is generally not possible to show that such distributions satisfy condition $\mathfrak{F}_{UM}$.

6.1. Cramér-Lundberg approximation

Suppose the arrival of claims filed against an insurance company follows a Poisson process $\{N(\tau) : \tau > 0\}$ with rate $\lambda > 0$. Let successive positive claim amounts be the i.i.d. absolutely continuous values $\{X_i\}$ with mean μ so the compound Poisson process $L(\tau) = \sum_{i=1}^{N(\tau)} X_i$ describes the company payout after time τ. Also, suppose the company's premiums increase revenues at constant rate $\sigma > 0$ with $\sigma > \lambda\mu$ so the premium rate exceeds the claim rate and $\rho = \lambda\mu/\sigma < 1$. If the company starts with initial reserve t, then the probability of eventual ruin for the company is $S(t) = P\{L(\tau) - \sigma\tau > t \quad \exists_{\tau > 0}\}$. With claims filed after interarrival times $\{T_i\}$, which are i.i.d. Exponential (λ), then the ruin must occur at an arrival epoch of a claim so that

$$S(t) = P\left\{ S_n = \sum_{i=1}^{n} (X_i - \sigma T_i) > t \quad \exists_{n \geq 1} \right\} = P(R > t) = S_R(t),$$

where $R = \sup_{n \geq 1} S_n$ is the maximum loss that occurs with claim $N = \arg\sup_{n \geq 1} S_n$. Denote the density of R as $f_R(t)$ so $-S'_R(t) = f_R(t)$ for a.e. t.

Suppose the claim amount X has MGF $\mathcal{M}_X$ which converges on $\{\mathrm{Re}(s) < c\}$ with $c > 0$. Let E have the excess life distribution for X with MGF $\mathcal{M}_E$. Then, R has MGF

$$\mathcal{M}_R(s) = \int_{0^-}^{\infty} e^{st} dF_R(t) = \frac{1 - \rho}{1 - \rho \mathcal{M}_E(s)} = (1 - \rho) + \rho \sum_{k=0}^{\infty} (1 - \rho)\rho^k \mathcal{M}_E^{k+1}(s) \qquad (6.1)$$

which converges on $\{\mathrm{Re}(s) < b\}$ where $b \in (0, c)$; see Tijms (2004, §8.4). Convergence bound b is a simple pole and results as a simple zero of $1 - \rho \mathcal{M}_E(s)$ since $\mathcal{M}_E(s)$ is strictly increasing. The rightmost expression in (6.1) reveals R as a mixture distribution with a point mass of $1 - \rho$ at $t = 0$ and an absolutely continuous component with mass ρ on $(0, \infty)$. The following result, based on Ikehara conditions, is shown in §B.4.1.

Theorem 3. *Suppose absolutely continuous claim amount X is as described above and satisfies uniform Ikehara condition $\mathfrak{I}_{ND(0,\infty)}$. Then, $R^+ = R|R > 0$, the positive and absolutely continuous portion of maximum loss R, satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{ND(0,\infty)}$. Thus,*

$$f_R(t) = \rho f_{R^+}(t) \sim \frac{b(1 - \rho)}{\lambda \mathcal{M}'_X(b)/\sigma - 1} e^{-bt} \qquad t \to \infty, \qquad (6.2)$$

so that $S_R(t) = \rho S_{R^+}(t) \sim f_R(t)/b$.

Theorem 3 is a new stronger form of the Cramér-Lundberg approximation since it provides an expansion for the density of R in (6.2) and not just $S_R(t)$ as traditionally given in, for example, Asmussen (2000, III.5 thm. 5.3). The conclusions of Theorem 3 are stronger but they also require the stronger assumption that X satisfies $\mathfrak{I}_{ND(0,\infty)}$. Theorem 3 is proved by using Tauberian theory whereas the traditional expansion for $S_R(t)$ is derived by using renewal theory as in Feller (1971, §§XI.6 and XII.5).

6.2. Sparre Andersen risk model

This model generalises the Cramér-Lundberg model so claims can arrive according to a general renewal process rather than as a Poisson process. Assume absolutely continuous interarrival times $\{T_i\}$ are i.i.d. with a MGF $\mathcal{M}_T(s)$ which converges on $\{\mathrm{Re}(s) < a\}$ for $a > 0$. For all other quantities, we use the notation from the previous subsection. Assume i.i.d. absolutely continuous claims $X_1, X_2, \ldots$ have a MGF $\mathcal{M}_X(s)$ convergent on $\{\mathrm{Re}(s) < c\}$ for $c > 0$, and that $E(X - \sigma T) < 0$ so the drift of random walk $\{S_n = \sum_{i=1}^{n} (X_i - \sigma T_i)\}$ is negative. Let $b \in (0, c)$ be the unique (real) positive root of $\mathcal{M}_{X-\sigma T}(s) = \mathcal{M}_X(s)\mathcal{M}_T(-\sigma s) = 1$.

Theorem 4. *Suppose the conditions on $\mathcal{M}_T(s)$ and $\mathcal{M}_X(s)$ above. If X satisfies uniform Ikehara condition $\mathfrak{I}_{ND(0,\infty)}$, then $R^+ = R|R > 0$ satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{ND(0,\infty)}$ and*

$$f_R(t) \sim \alpha e^{-bt} \qquad t \to \infty, \tag{6.3}$$

so that $S_R(t) \sim \alpha e^{-bt}/b$. Here, $\alpha > 0$ is given in (8.54) of §B.4.2 as the negative residue of $\mathcal{M}_R(s)$ at b.

The expansion for density $f_R(t)$ in (6.3) is new and requires the additional assumption that X satisfies $\mathfrak{I}_{ND(0,\infty)}$. Expansion (6.3) ensures that the established expansion for survival function $S_R(t)$ also holds as previously given, for example, in Theorem 3.1, case (i), of Embrechts and Veraverbeke (1982). Note that the conditions on X in Theorem 4 are exactly the same as those used in Theorem 3 for the Cramér-Lundberg setting.

The rather long proof of Theorem 4 is given in §B.4.2 and demonstrates that R^+ satisfies the Ikehara conditions of Proposition 1. That $\mathcal{M}_{R^+}$ satisfies condition $\mathfrak{I}_{\mathcal{M}}$ follows from the Wiener-Hopf factorization that determines $\mathcal{M}_R$. To show condition $\mathfrak{I}_{UND}$, we use the compound geometric sum characterization of R in which $R = \sum_{i=0}^{N_+} L_i^+$ where $\{L_i^+ : i \geq 1\}$ are i.i.d. with the ascending ladder distribution of the random walk $\{S_n\}$ and L_0^+ denotes a point mass at 0. From this compound geometric sum, we determine that R^+ satisfies $\mathfrak{I}_{ND(0,\infty)}$ if L_1^+ satisfies $\mathfrak{I}_{ND(0,\infty)}$ and this holds if X satisfies $\mathfrak{I}_{ND(0,\infty)}$. The latter result follows by noting that the last convolved step amount (or $Y = X - \sigma T$ addend) in the ladder height L_1^+ is necessarily a positive step which depends on the X contribution in the step amount $X - \sigma T$.

Feller conditions do not suffice in this line of proof. Tilting parameter b must be used to show R^+ satisfies $\mathfrak{F}_{UM}$ whereas the assumption that X satisfies $\mathfrak{F}_{UM}$ uses the larger tilting parameter $c > b$. Thus, the fact that X satisfies $\mathfrak{F}_{UM}$ has no bearing on whether R^+ can satisfy $\mathfrak{F}_{UM}$.

6.3. More general compound distributions

In the two previous subsections, the ruin amount $R = \sum_{k=0}^N X_k$ assumes that N has a geometric mass function, and this results in an infinite mixture distribution with geometric weights. We now give an expansion for the density of R when N assumes a more general mass function with probability generating function (PGF) $\mathcal{P}(z) = \sum_{k=0}^\infty p(k)z^k$ that converges on $\{|z| < r\}$ for $r > 1$. Suppose X_0 has a point mass at zero and let $\{X_k : k \geq 1\}$ be i.i.d. absolutely continuous positive claim amounts with MGF $\mathcal{M}(s)$ that converges on $\{\mathrm{Re}(s) < c\}$ or $\{\mathrm{Re}(s) \leq c\}$ for $c > 0$. Now, R has a compound distribution with MGF $\mathcal{P}\{\mathcal{M}(s)\}$.

The next result shows that the density of R has a gamma-like tail under two conditions: $\mathcal{M}$ satisfies Ikehara condition $\mathfrak{I}_{ND(0,\infty)}$, and $\mathcal{P}$ satisfies Darboux condition $\mathfrak{D}_{\mathcal{M}}$.

Condition $\mathfrak{D}_{\mathcal{M}}$ holds if convergence bound $r > 1$ for $\mathcal{P}$ is the only singularity on the circle $\{z \in C : |z| = r\}$ and $\mathcal{P}(z) = g(z)(r-z)^{-w} + h(z)$, where $w > 0$, $g(z)$ and $h(z)$ are analytic on the closed disc $\{|z| \leq r\}$, and $g(r) \neq 0$. See §B.4.3 for a proof.

Theorem 5. *Let $\mathcal{P}(z)$ and $\mathcal{M}(s)$ be as described above and suppose $r < \mathcal{M}(c) \leq \infty$. Also suppose $\mathcal{M}$ satisfies uniform Ikehara condition $\mathfrak{I}_{ND(0,\infty)}$ and $\mathcal{P}$ satisfies Darboux condition $\mathfrak{D}_{\mathcal{M}}$. Then, $R^+ = R|R > 0$ satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{ND(0,\infty)}$, R has limiting hazard rate $b \in (0, c)$, where b is the unique root of $\mathcal{M}(s) = r$ in $(0, c)$, and*

$$f_R(t) \sim \frac{g(r)}{\Gamma(w)\{\mathcal{M}'(b)\}^w} t^{w-1} e^{-bt} \qquad t \to \infty. \tag{6.4}$$

Consequently, $S_R(t) \sim f_R(t)/b$ and, for the stop-loss premium, $\int_t^\infty S_R(u)du \sim f_R(t)/b^2$ as $t \to \infty$.

The density expansion is new while the expansion $S_R(t) \sim f_R(t)/b$ replicates the result of Embrechts et al. (1985) who assume $p(k) \sim C_1 k^{w-1} r^{-k}$ as $k \to \infty$ so that N has a Negative Binomial $(w, 1/r)$ tail. Note that the Darboux condition on $\mathcal{P}$ is slightly stronger, and, by Proposition 2, implies that the expansion for $p(k)$ holds with $C_1 = g(r)r^{-w}/\Gamma(w)$. Thus, Theorem 5 uses a slightly stronger condition but also returns a stronger result by giving the density expansion in (6.4). In addition, the conditions of Theorem 5 are more useful since the easiest way to justify the expansion for $p(k)$ is to verify the Darboux assumption on $\mathcal{P}$.

Example 6. (Negative Binomial (w, p) weights). This example is of practical importance since such weights represent over-dispersed Poisson weights that result from a gamma mixture of Poisson weights. Let N have mass function $p(k) = \binom{w+k-1}{k} p^w q^k$ on $k \geq 0$, with $q = 1 - p$, so that $\mathcal{P}(z) = p^w/(1 - qz)^w$ for $z < 1/q$. Then, R has the density expansion in (6.4) where $g(r) = (p/q)^w$ and b solves $\mathcal{M}(b) = 1/q$. The expansion for $S_R(t)$ was originally derived in Sundt (1982) under the assumption that $e^{bt} S_R(t)$ is ultimately monotone.

A version of Theorem 5 holds with integer-valued non-negative claim amounts $\{X_k : k \geq 1\}$ which are i.i.d. with MGF $\mathcal{M}(s) = \mathcal{P}_X(e^s)$. The proof is given in §B.4.3.

Corollary 6. *Suppose claim amount X is non-negative and integer-valued and let $\mathcal{P}(z)$ and $\mathcal{M}(s) = \mathcal{P}_X(e^s)$ have the same properties as in Theorem 5 (except now X does not satisfy $\mathfrak{I}_{ND(0,\infty)}$). Then compound sum $R = \sum_{k=0}^N X_k$ has limiting hazard rate $1 - e^{-b}$, where $b \in (0, c)$ is the unique root of $\mathcal{P}_X(e^s) = r$ in $(0, c)$. Furthermore,*

$$p_R(n) \sim \frac{g(r)e^{-bw}}{\Gamma(w)\{\mathcal{P}'_X(e^b)\}^w} n^{w-1} e^{-bn} \qquad n \to \infty, \tag{6.5}$$

$S_R(n) \sim (1-e^{-b})^{-1}p_R(n)$, *and stop-loss premium,* $\sum_{k=n+1}^{\infty} S_R(k) \sim (1-e^{-b})^{-2}p_R(n+1)$
as $n \to \infty$.

These expansions agree with those in Willmot (1989) who assumes that N has mass function $p(k) \sim C_1 k^{w-1} r^{-k}$ as $k \to \infty$; note this is implied by the Darboux condition on $\mathcal{P}$.

6.3.1. *Compound distributions with multiple claim distributions and multivariate weights*

Compound distributions are generalised to allow M distinct categories of positive claim amounts $X_1, \ldots, X_M$. Suppose $\{X_i\}$ are absolutely continuous and independent, and X_i has MGF $\mathcal{M}_i(s)$ which converges on $\{\mathrm{Re}(s) < c_i\}$ or $\{\mathrm{Re}(s) \le c_i\}$ for $c_i > 0$. If $\{X_{ij} : j \ge 1\}$ are i.i.d. absolutely continuous claims from category i, then $R = \sum_{i=1}^{M} 1\{N_i \ge 1\} \sum_{j=1}^{N_i} X_{ij}$ denotes a compound variable over the M categories of claims, where $\mathbf{N} = (N_1, \ldots, N_M)^T$ tallies counts for the various categories. Let the components of $\mathbf{N}$ have a general distribution with multivariate PGF $\mathcal{P}(\mathbf{z})$, where $\mathbf{z} = (z_1, \ldots, z_M)^T \in \Re^M$. Suppose $\mathcal{P}$ converges on maximal set $\mathcal{O} \supset \{\mathbf{z} \in \Re^M : |z_i| \le 1 \text{ for } i = 1, \ldots, M\}$ so as to avoid heavy-tailed counting components. Also, let $\mathcal{M}(s) = \{\mathcal{M}_1(s), \ldots, \mathcal{M}_M(s)\}^T$, with scalar $s \in \Re$, be the vector of MGFs for $\mathbf{X} = (X_1, \ldots, X_M)^T$. Then, R has MGF $\mathcal{P}\{\mathcal{M}(s)\}$ which converges in a neighbourhood of 0. The assumption that Ikehara condition $\mathfrak{I}_{ND(0,\infty)}$ holds for the components of $\mathcal{M}(s)$ can be used to show that $f_R(t)$ and $S_R(t)$ have Gamma tails with a limiting hazard rate. A proof is given in §B.4.3.

Theorem 6. *Let* $\mathcal{P}(\mathbf{z})$ *and* $\mathcal{M}(s)$ *be as described above and suppose each component of* $\mathcal{M}(s)$ *satisfies* $\mathfrak{I}_{ND(0,\infty)}$. *Let* $\mathcal{P}(\mathbf{z}) = \mathcal{N}(\mathbf{z})/\mathcal{D}(\mathbf{z})$ *have maximal convergence region* $\mathcal{O}$, *where* $\{\mathbf{z} : |z_i| \le 1 \text{ for } i = 1, \ldots, M\} \subset \mathcal{O} \subseteq \{\mathbf{z} \in \Re^M : \mathcal{D}(\mathbf{z}) > 0\}$. *Take* $c_* = \min_i c_i$ *and suppose* $\mathcal{D}\{\mathcal{M}(s)\} = 0$ *admits a smallest positive root* $b \in (0, c_*)$ *which is an m-zero. Let* $\mathcal{N}(\mathbf{z})$ *and* $\mathcal{D}(\mathbf{z})$ *be analytic at* $\mathcal{M}(b) \in \Re^M$ *with* $\mathcal{N}\{\mathcal{M}(b)\} \ne 0$. *Then,* $R^+ = R \,|\, R > 0$ *satisfies* $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{ND(0,\infty)}$ *and*

$$f_R(t) \sim \beta t^{m-1} e^{-bt} \qquad S_R(t) \sim b^{-1} f_R(t) \qquad t \to \infty, \tag{6.6}$$

with

$$\beta = \frac{1}{(m-1)!} \lim_{s \to b} (b-s)^m \mathcal{P}\{\mathcal{M}(s)\} = \frac{(-1)^m m \mathcal{N}\{\mathcal{M}(b)\}}{\partial^m \mathcal{D}\{\mathcal{M}(s)\}/\partial s^m|_{s=b}}. \tag{6.7}$$

An important feature of Theorem 6 is that it lacks the assumption that b is a dominant pole for $\mathcal{P}\{\mathcal{M}(s)\}$; this emerges subject to the quite minimal conditions placed upon $\mathcal{P}$.

The important applications for Theorem 6 are deferred to the next section since they are rather involved and concern the derivation of exponential tail behaviour for the

densities and survival functions of first-passage distributions in finite-state continuous-time semi-Markov processes. For now, we consider two simpler examples.

Example 7. (Independent counts). Let $\mathbf{N}$ have independent components, so $\mathcal{P}(\mathbf{z}) = \prod_{i=1}^{M} \mathcal{P}_i(z_i)$ and $R = \sum_{i=1}^{M} R_i$ is a sum of independent compound sum variables with $R_i = 1\{N_i \geq 1\} \sum_{j=1}^{N_i} X_{ij}$ having MGF $\mathcal{P}_i\{\mathcal{M}_i(s)\}$ that converges on $\text{Re}(s) < b_i > 0$. For simplicity, suppose $b_1 < \min_{i \geq 2} b_i$ so R_1 is the strongest term of R. For the setting in which $\mathcal{P}_1\{\mathcal{M}_1(s)\}$ admits a pole of order w_1 at $b_1 > 0$, then Theorem 6 applies by taking $\mathcal{D}(\mathbf{z}) = 1/\mathcal{P}_1(z_1)$ and $\mathcal{N}(\mathbf{z}) = \prod_{i=2}^{M} \mathcal{P}_i(z_i)$. In this case, $\mathcal{M}_1(b_1) = r_1 > 1$, the radius of convergence for $\mathcal{P}_1$, and

$$f_R(t) \sim f_{R_1}(t) \prod_{i=2}^{M} \mathcal{P}_i\{\mathcal{M}_i(b_1)\} \sim \beta t^{w_1-1} e^{-b_1 t} \qquad t \to \infty, \tag{6.8}$$

where

$$\beta = \frac{g_1(r_1)}{\Gamma(w_1)\{\mathcal{M}_1'(b_1)\}^{w_1}} \prod_{i=2}^{M} \mathcal{P}_i\{\mathcal{M}_i(b_1)\}$$

and $g_1(r_1) = \lim_{z_1 \to r_1}(r_1 - z_1)^{w_1} \mathcal{P}_1(z_1)$.

The expansion in (6.8) also holds when b_1 is a singularity of order $w_1 > 0$ and not a pole. The argument for this requires expressing R as a finite mixture of 2^M terms, where terms are determined by which components in $\{R_i\}$ are positive and which ones are point masses at 0. There are 2^{M-1} dominant terms each convolving the dominant component $R_1^+ = R_1 | R_1 > 0$. Expansion (6.8) results when Corollary 2 of §5.1 is applied to these mixture terms and Theorem 5 is applied to R_1. See §B.4.3 for details.

Example 8. (Multivariate Negative Binomial $(m, \mathbf{p})$ weights). Consider $M + 1$ claim categories with claims sampled as independent multivariate Bernoulli trials in which category i has probability p_i, and the components of $\mathbf{p} = (p_1, \ldots, p_M, p_{M+1})$ sum to 1. If sampling stops on the mth occurrence of category $M + 1$, then $\mathbf{N}$, the count totals for the first M categories, has PGF

$$\mathcal{P}(\mathbf{z}) = p_{M+1}^m \left(1 - \textstyle\sum_{i=1}^{M} p_i z_i\right)^{-m}.$$

Taking $\mathcal{M}(s) = \{\mathcal{M}_1(s), \ldots, \mathcal{M}_M(s)\}$, then the claim over the first M categories is R with MGF $\mathcal{P}\{\mathcal{M}(s)\}$ and the total claim is $R_T = R + \sum_{j=1}^{m} X_{M+1,j}$ with MGF $\mathcal{P}\{\mathcal{M}(s)\}\mathcal{M}_{M+1}^m(s)$. For simplicity, we assume that R is stronger than $\{X_{M+1,j}\}$. Subject to the components of $\mathcal{M}(s)$ satisfying $\mathfrak{I}_{ND(0,\infty)}$, then direct application of Theorem 6 gives

$$f_{R_T}(t) \sim \frac{\gamma^m}{(m-1)!} t^{m-1} e^{-bt} \mathcal{M}_{M+1}^m(b) \qquad t \to \infty, \tag{6.9}$$

where b is the smallest positive root of $1-\sum_{i=1}^{M} p_i \mathcal{M}_i(s) = 0$ and $\gamma = p_{M+1}/\{\sum_{i=1}^{M} p_i \mathcal{M}'_i(b)\}$.

When claim amounts $X_1, \ldots, X_M$ are non-negative and integer-valued, then subject to the same conditions as in Theorem 6, the compound distribution R has a negative binomial tail; see §B.4.3 for a proof.

Theorem 7. *Suppose all the assumptions in Theorem 6 but now let $X_1, \ldots, X_M$ be non-negative and integer-valued (so components of $\mathcal{M}(s)$ need not satisfy $\mathfrak{I}_{ND(0,\infty)}$). Then,*

$$p_R(n) \sim \beta n^{m-1} e^{-bn} \qquad S_R(t) \sim \frac{1}{1-e^{-b}} \, p_R(n) \qquad n \to \infty, \qquad (6.10)$$

where β is given by the rightmost expression in (6.7).

7. First-passage distributions for semi-Markov processes

Consider a semi-Markov process (SMP) with $M < \infty$ states. Under relatively mild conditions, we show that first-passage distributions from one state to another state have limiting hazard rates and exponential-like or geometric-like tails. Such results are derived by characterizing first-passage distributions as compound sums as described in Proposition 3. Then, for continuous-time processes, exponential-like tails follow from Theorem 6 under Ikehara $\mathfrak{I}_{ND(0,\infty)}$ assumptions, and, for integer-time processes, geometric-like tails follow from Theorem 7 under the minimal conditions afforded by Darboux's theorem.

Let $\mathcal{S} = \{1, \ldots, M\}$ be the states of a continuous- or integer-time SMP and consider a sojourn from state 1 to M. The SMP is characterised by its $M \times M$ kernel matrix $\mathbf{K}(t) = \{p_{ij} G_{ij}(t) : i, j \in \mathcal{S}\}$, where $\mathbf{P} = \{p_{ij}\}$ is the transition probability matrix of the associated jump chain for state transitions, and G_{ij} is the holding time distribution in state i given state j is certain to be the next destination. The $M \times M$ Laplace-Stieltjes transform of $\mathbf{K}(t)$ also characterises the SMP and is given by

$$\mathbf{T}(s) = \int_0^\infty e^{st} d\mathbf{K}(t) = \{p_{ij} \mathcal{M}_{ij}(s)\} = \mathbf{P} \odot \mathcal{M}(s),$$

where $\mathcal{M}(s) = \{\mathcal{M}_{ij}(s)\}$ is $M \times M$, $\mathcal{M}_{ij}(s)$ is the MGF of $G_{ij}(t)$ and convergent on $\{\mathrm{Re}(s) < c_{ij}\}$ or $\{\mathrm{Re}(s) \leq c_{ij}\}$, and $\odot$ denotes a Hadamard product. Matrix $\mathbf{T}(s)$ is called the transmittance matrix since its entries consist of transmittances defined as a probability $\times$ a MGF.

If X is the first-passage time from $1 \to M$, then it has a potentially defective distribution in which $f_{1M} = P(X < \infty) \in (0,1]$ and $\mathcal{F}_{1M}(s) = E\{e^{sX} | X < \infty\}$ is the

MGF given a finite sojourn. The product of these two quantities determines the first-passage transmittance associated with the sojourn or $f_{1M}\mathcal{F}_{1M}(s) = E\{e^{sX}1_{(X<\infty)}\}$. Butler (2000) has shown that this first-passage transmittance takes the following simple form in terms of matrix $\mathbf{T}(s)$:

$$f_{1M}\mathcal{F}_{1M}(s) = \frac{(M,1)\text{-cofactor of }\{\mathbf{I}_M - \mathbf{T}(s)\}}{(M,M)\text{-cofactor of }\{\mathbf{I}_M - \mathbf{T}(s)\}} = \frac{(-1)^{M+1}|\mathbf{\Psi}_{M1}(s)|}{|\mathbf{\Psi}_{MM}(s)|} \tag{7.1}$$

where $\mathbf{\Psi}_{ij}(s)$ is the (i,j)th minor of $\mathbf{I}_M - \mathbf{T}(s)$, or the submatrix of $\mathbf{I}_M - \mathbf{T}(s)$ with the ith row and jth column removed. In either continuous or integer time, the ratio (7.1) has a maximal convergence region that contains 0 of the form $\{\operatorname{Re} s\} < b\}$ or $\{\operatorname{Re} s \leq b\}$ for some $b > 0$ under these conditions:

(i) The system states $\mathcal{S}$ consist of exactly those states that are relevant to passage from $1 \to M$ and contain no non-relevant states, i.e. states that are not accessible while completing a sojourn from $1 \to M$. (Thus, all row sums for $\mathbf{P}$ may not be 1; see below.)

Non-relevant states include absorbing (classes of) states other than M and perhaps some transient states that are not accessible during the sojourn. To determine $\mathcal{S}$ and hence $\mathbf{P}$, start with all states, both relevant and non-relevant, so transition probability rows all sum to 1. Now, delete the rows and columns associated with all non-relevant states; if absorbing (classes of) states have been removed then some row $i \in \mathcal{S}\backslash\{M\}$ of $\mathbf{P}$ will not sum to 1. If $p_{i\cdot} = \sum_{j\in\mathcal{S}} p_{ij} < 1$, then the jump chain $\mathbf{P}$ may pass from i into an absorbing (class of) state(s) other than M w.p. $1 - p_{i\cdot}$. When such occurs, the sojourn time is ∞, $f_{1M} < 1$, and the first-passage distribution is defective.

(ii) The convergence regions for the MGFs in the first $M - 1$ rows of $\mathbf{T}(s)$, which are those used in the ratio (7.1), include an open neighbourhood of 0, i.e. $0 < c_* = \min\{c_{ij} : i \in \mathcal{S}\backslash\{M\}; j \in \mathcal{S}\}$.

We now show that MGF $\mathcal{F}_{1M}$ represents a compound distribution. During a first-passage sojourn from $1 \to M$, let N_{ij} count the number of $i \to j$ transitions of the jump chain and denote $\mathbf{N} = \{N_{ij}\}$ as $M \times M$. Now, let $\mathcal{P}(\mathbf{Z}|Y < \infty)$ be the conditional PGF of $\mathbf{N}$ in the $M \times M$ variables of matrix $\mathbf{Z} = \{z_{ij}\}$ given $Y = \sum_{i,j=1}^{M} N_{ij} < \infty$, where Y counts the total number of steps required for first passage. The first-passage time MGF is $\mathcal{F}_{1M}(s) = \mathcal{P}\{\mathcal{M}(s)|Y < \infty\}$ as stated below and shown in §B.5.1.

Proposition 3. *Assuming (i), the conditional probability generating function for* $\mathbf{N}$ *given* $Y < \infty$ *is*

$$\mathcal{P}(\mathbf{Z}|Y < \infty) = \frac{1}{f_{1M}}\frac{(M,1)\text{-cofactor of }\{\mathbf{I}_M - \mathbf{P}\odot\mathbf{Z}\}}{(M,M)\text{-cofactor of }\{\mathbf{I}_M - \mathbf{P}\odot\mathbf{Z}\}}. \tag{7.2}$$

The convergence in (7.2) is on $\{\mathbf{Z} : |\lambda_1\{(\mathbf{P}\odot\mathbf{Z})_{\backslash MM}\}| < 1\} \supset \{\mathbf{Z} : |z_{ij}| \leq 1$ *for* $i, j = 1, \ldots M\}$, *where* $\lambda_1\{\cdot\}$ *denotes the eigenvalue of largest modulus for the matrix*

argument, and $(\mathbf{P} \odot \mathbf{Z})_{\backslash MM}$ *is* $\mathbf{P} \odot \mathbf{Z}$ *without its* M*th row and* M*th column. First-passage sojourn time* X*, when* $X < \infty$*, is a compound distribution of the form*

$$X = \sum_{i=1}^{M-1} \sum_{j=1}^{M} 1\{N_{ij} \geq 1\} \sum_{k=1}^{N_{ij}} X_{ijk} \tag{7.3}$$

where $\{X_{ijk} : k \geq 1\}$ *are i.i.d.* $G_{ij}(t)$*. Based upon this, the conditional MGF of* $X | X < \infty$ *is* $\mathcal{F}_{1M}(s) = \mathcal{P}\{\mathcal{M}(s) | Y < \infty\}$ *as given in* (7.1).

In this first-passage characterization, exit from state M is counted as $N_{Mj} = 0$ w.p. 1 for $j \geq 1$ and its PGF is the value 1 and degenerate. This is consistent with (7.3), which does not involve $\{N_{Mj}\}$, and also (7.2) whose cofactors do not depend on the Mth row of $\mathbf{P} \odot \mathbf{Z}$.

Proposition 3 makes two important points. First, it provides another formal derivation of the identity in (7.1) thus confirming the initial proof in Butler (2000). Secondly, and most crucially for our purposes, it characterises the first-passage distribution as a compound distribution to which we can apply Theorem 6. We now make some additional assumptions that are needed for using Theorem 6 in the continuous-time SMP setting. These additional assumptions can be verified as holding in all the various practical examples that have been considered in Butler (2000, 2007 ch. 13) and in the additional references therein.

(iii) Assume convergence bound $b \in (0, c_*)$ for $\mathcal{F}_{1M}(s)$ is a simple pole that results as the smallest positive zero of $|\mathbf{\Psi}_{MM}(s)|$ with $|\mathbf{\Psi}_{M1}(b)| \neq 0$.

(iv) Suppose the first $M-1$ rows of $\mathbf{K}(t)$ consist of absolutely continuous component distributions. Define $\mathcal{B} \subset \mathcal{S} \times \mathcal{S}$ as a *blockade* of state transitions for the sojourn $1 \to M$ if all paths from $1 \to M$ must incur at least one state transition in $\mathcal{B}$. Assume there exists a blockade $\mathcal{B}$ such that each blockade member $(i, j) \in \mathcal{B}$ has a density $g_{ij}(t)$ which satisfies $\mathfrak{I}_{ND(0,\infty)}$.

Theorem 8. *Suppose a continuous-time semi-Markov process satisfies conditions (i)–(iv) above. Then the first-passage time distribution of* $X | X < \infty$ *from* $1 \to M$ *has asymptotic hazard rate* $b > 0$*, as given in (iii), with density and survival functions*

$$f(t) \sim \beta e^{-bt} \qquad and \qquad S(t) \sim \frac{1}{b} f(t) \qquad t \to \infty,$$

where

$$\beta = -\operatorname{Res}\{\mathcal{F}_{1M}(s); b\} = -\frac{|\mathbf{\Psi}_{M1}(b)|}{|\mathbf{\Psi}_{M1}(0)|} \frac{|\mathbf{\Psi}_{MM}(0)|}{\operatorname{tr}[\operatorname{adj}\{\mathbf{\Psi}_{MM}(b)\} \dot{\mathbf{\Psi}}_{MM}(b)]}, \tag{7.4}$$

adj$\{\cdot\}$ *denotes the* $(M-1) \times (M-1)$ *adjoint of the matrix argument, and* $\dot{\mathbf{\Psi}}_{MM}(b) = d\mathbf{\Psi}_{MM}(s)/ds|_{s=b}$ *is* $(M-1) \times (M-1)$.

Proof. Under the conditions of Theorem 8, the compound distribution for $X \,|\, X < \infty$, as characterised in Proposition 3, satisfies the conditions of Theorem 6; see §B.5.2 for details. □

It is noteworthy that Theorem 8 lacks the full Ikehara assumption $\mathfrak{I}_{\mathcal{M}}$. Assumption (iii) stipulates that b is a simple pole for $\mathcal{F}_{1M}$ but it does not require that b be a dominant pole as also stipulated in $\mathfrak{I}_{\mathcal{M}}$. This latter fact emerges as a consequence of the method of proof in which Theorem 6 is applied to the compound distribution of $\mathcal{F}_{1M}$ as described in Proposition 3.

Example 9. (Cramér-Lundberg and Sparre Andersen). In both of these models, the conditional distribution for positive ruin $R^+ = R|R > 0$ can be considered as an example of a first-passage distribution for a certain SMP. For the Cramér-Lundberg model, the two state SMP with transmittance matrix

$$\mathbf{T}(s) = \left(\begin{array}{cc} \rho \mathcal{M}_E(s) & (1-\rho)\mathcal{M}_E(s) \\ 0 & 0 \end{array} \right) \tag{7.5}$$

has first-passage transmittance from $1 \rightarrow 2$ computed from (7.1) as $f_{12}\mathcal{F}_{12}(s) = (1 - \rho)\mathcal{M}_E(s)/\{1 - \rho\mathcal{M}_E(s)\}$, which is $\mathcal{M}_{R^+}(s)$ computed from the rightmost summation component in (6.1). For the Sparre Andersen model, if the first row entries in (7.5) are $e^{-B}\mathcal{M}_{L^+}(s)$ and $(1 - e^{-B})\mathcal{M}_{L^+}(s)$, then $f_{12}\mathcal{F}_{12}(s)$ yields $\mathcal{M}_{R^+}$ as given in the middle expression for $\mathcal{M}_{R^+}$ in (8.47) of §B.4.2. The conditions and proofs used in Theorems 3 and 4 are needed to ensure that the associated SMPs satisfy conditions (i)–(iv) of Theorem 8. In particular, the densities for excess life E and the ascending ladder variable L^+ must satisfy $\mathfrak{I}_{ND(0,\infty)}$ so condition (iv) is satisfied. Thus, the conclusions of Theorems 3 and 4 follow as special cases of Theorem 8 applied to simple SMPs as in (7.5).

Example 10. (GI/M/1 and M/G/1 queues). The first passage time from an empty queue (state 0) to queue length M for either of these queues is a passage time for a SMP; see Butler (2000, §6 and 2007, §13.2.5) and Butler and Huzurbazar (2000, §7) respectively. In either setting, it can be shown that all entries of the $(M + 1) \times (M + 1)$ kernel matrix $\mathbf{K}(t)$ satisfy $\mathfrak{I}_{ND(0,\infty)}$ (so condition (iv) is satisfied) when the interarrival distribution of the renewal process satisfies $\mathfrak{I}_{ND(0,\infty)}$. Three such interarrival distributions are the particular Gamma, compound Poisson, and inverse Gaussian distributions used as numerical examples in the references above. Indeed, these examples lead to transmittance matrices with entries that also satisfy the remaining conditions (i)–(iii) so that Theorem 8 applies. Thus, the true hazard functions approach an asymptote with value b as suggested in the plots of saddlepoint approximations for such hazard functions in Butler (2000, §6 and 2007, §13.2.5). Theorem 8 also proves that the survival functions for these examples have exponential orders $\beta e^{-bt}/b$. This is also suggested in the additional plots of saddlepoint approximations for these survival functions.

The analogous integer-time result follows directly from Theorem 7 and is based upon the minimal conditions of Proposition 2. Having proven Proposition 2 by using Darboux's theorem rather than Theorem 5 in Feller (1971, XIII.5), we avoid the need to assume that the sojourn mass function is ultimately non-decreasing as required in Feller's theorem. Making such an assumption would be rather pointless as it requires additional and mostly unverifiable knowledge about the very function for which we are providing an asymptotic expansion.

Theorem 9. *Suppose an integer-time semi-Markov process satisfies conditions (i)– (iii). Then the first-passage time distribution of $X \mid X < \infty$ from $1 \to M$ has asymptotic hazard rate $1 - e^{-b}$, with b given in (iii), and mass and survival functions*

$$p(n) \sim \beta e^{-bn} \qquad \text{and} \qquad S(n) \sim \frac{1}{1 - e^{-b}}\, p(n) \qquad n \to \infty,$$

where β is given in (7.4).

Theorem 9 lacks the full Darboux $\mathfrak{D}_{\mathcal{M}}$ assumption since condition (iii) only assumes that b is a simple pole for $\mathcal{F}_{1M}$. The requirement that it also be a dominant pole emerges as a consequence of Theorem 7.

In the Markov setting, the results of these two theorems are, of course, well-known as they simply state the known asymptotic behaviour of the phase distributions which represent the sojourn times. Also, for SMPs in which $\mathbf{T}(s)$ is composed of rational MGFs, the results follow directly from partial fraction expansion of $\mathcal{F}_{1M}(s)$ based upon (7.1). The importance of these two theorems, however, is not that they apply to such rational settings, but rather that they apply to non-rational settings in which the entries of $\mathbf{T}(s)$ may be non-rational MGFs as occurs in the broader class of SMPs. In such non-rational settings, the theorems show that exponential-like/geometric-like tails result quite generally for first-passage distributions of SMPs as they do for the more restrictive class of Markov processes. Such conclusions reinforce the *insensitivity* properties of SMPs discussed by Tijms (2003, §5.4) in which, for large t or n, SMPs behave much like Markov processes and exhibit insensitivity to the actual non-exponential/non-geometric holding time distributions used in the kernel $\mathbf{K}(t)$.

Exponential-like/geometric-like tails can also be shown to hold for other types of sojourns in finite state SMPs, such as first return to a single state or first-passage from one state to a subset of states. These results make use of other first-passage transmittances as given in Theorems 2 and 3 of Butler (2000) or Butler (2007, §§13.2.6 and 13.3). Details of this will be presented elsewhere.

Assumption (iii), forming part of the Ikehara $\mathfrak{I}_{\mathcal{M}}$ condition, is most likely unnecessary for the conclusions of Theorems 8 and 9 to hold and can be replaced by weaker assumptions concerning the composition of states in $\mathcal{S}$. For example, if $\mathcal{S} \backslash \{M\}$ consists of states

that all communicate, then in Markov processes $b \in (0, c_*)$ is a simple pole, which is a result that follows from the associated Perron-Frobenius theory. Similar results should apply to SMPs and will be addressed in future work. Furthermore, assumption (iv), used for continuous-time processes, can likely be replaced with alternative integrability assumptions on the components of $\mathbf{T}(s)$. However, even with the potential for relaxing some of these assumptions, Theorems 8 and 9 are the first formal results of their kind for general SMPs and apply to a very broad class of SMPs used in many classical applications of stochastic modelling and multi-state survival analysis.

8. Logarithmic singularities

Asymptotic expressions are given for distributions whose boundary singularity b is logarithmic. Proofs are given in §A.6.

Proposition 4. *Let X have an absolutely continuous distribution on $(0, \infty)$ and moment generating function $\mathcal{M}(s)$ which converges on $\{s \in C : \mathrm{Re}(s) < b\}$ for $b > 0$. If X satisfies condition $\mathfrak{I}_{UND}$ of Proposition 1 and condition $\mathfrak{L}_{\mathcal{M}}$ below, then X has limiting hazard b with*

$$f(t) \sim g_m(b)m(\ln t)^{m-1}t^{-1}e^{-bt} \quad \text{and} \quad S(t) \sim f(t)/b \qquad t \to \infty. \tag{8.1}$$

($\mathfrak{L}_{\mathcal{M}}$) b is a logarithmic singularity for $\mathcal{M}(s)$ of the form

$$\mathcal{M}(s) = \sum_{j=1}^{m} g_j(s) \left\{ -\ln(b-s) \right\}^j + h(s), \tag{8.2}$$

where $\{g_j(s)\}$ and $h(s)$ are analytic on $\{s \in C : \mathrm{Re}(s) \le b\}$ and $g_m(b) \ne 0$.

Example 11. The exponential integral function $E_1(z) = \int_z^\infty t^{-1}e^{-t}dt$ defines the density

$$f(t) = E_1(1)^{-1}t^{-1}e^{-t} \qquad t > 1 \tag{8.3}$$

with MGF $\mathcal{M}(s) = E_1(1-s)/E_1(1)$ which converges on $\{s \in C : \mathrm{Re}(s) < 1\}$. Simple computations show that $\mathfrak{I}_{ND(0,\infty)}$ holds for any tilting parameter exceeding 1. Condition $\mathfrak{L}_{\mathcal{M}}$ holds due to the relationship of $E_1(1-s)$ to $-\ln(1-s)$, given in Abramowitz and Stegun (1972, eqn. 5.1.11, p. 229), in which $\mathcal{M}(s) = -E_1(1)^{-1}\ln(1-s) + h(s)$, for h analytic on $\{s \in C\}$. From Proposition 4, the asymptotic hazard is 1 and the order of $f(t)$ in (8.1) agrees with the exact density in (8.3) with $m = 1$ and $g_1(1) = E_1(1)^{-1}$.

If X_m is a convolution of m such i.i.d. variables, then MGF $\mathcal{M}(s)^m$ can be expanded using the binomial theorem to show it has form (8.2) with $g_m(s) = E_1(1)^{-m}$. The density

for X_m satisfies $\mathfrak{I}_{ND(0,\infty)}$ as shown by repeatedly using the convolution argument of Lemma B1 in §B.3.1 starting with X_1. Thus, the asymptotic hazard is 1 and

$$f_{X_m}(t) \sim E_1(1)^{-m} m (\ln t)^{m-1} t^{-1} e^{-t} \qquad t \to \infty. \tag{8.4}$$

For the case $m = 2$, the expansion in (8.4) is confirmed by direct computation in §A.6.

Similar expansions hold for lattice distributions with a log-singularity as in (8.6). A proof is given in §A.6.

Proposition 5. *Suppose X has non-negative integer support and moment generating function $\mathcal{M}(s)$ that converges on $\{s \in C : \operatorname{Re}(s) < b\}$, for $b > 0$. If X satisfies condition $\mathfrak{DL}_\mathcal{M}$ below, then X has limiting hazard $1 - e^{-b}$, with*

$$p(n) \sim g_m(e^b) m (\ln n)^{m-1} n^{-1} e^{-bn} \qquad n \to \infty, \tag{8.5}$$

and $S(n) \sim (1 - e^{-b})^{-1} p(n)$.

($\mathfrak{DL}_\mathcal{M}$) b is a singularity for $\mathcal{M}(s)$ which has the form

$$\mathcal{M}(s) = \sum_{j=1}^{m} g_j\left(e^s\right) \left\{-\ln(e^b - e^s)\right\}^j + h(e^s), \tag{8.6}$$

where $\{g_j(e^s)\}$ and $h(e^s)$ are analytic on $\{s \in C : \operatorname{Re}(s) \le b\}$, and $g_m(e^b) \ne 0$.

Example 12. The Logarithmic Series (p) distribution, with $p = 1 - q \in (0, 1)$, has mass function

$$p(n) = n^{-1} p^n / (-\ln q) \qquad n \ge 1 \tag{8.7}$$

and MGF $\mathcal{M}(s) = \ln(1 - pe^s)/\ln q$, which converges for $\operatorname{Re}(s) < b = -\ln p$. Proposition 5 determines the asymptotic hazard as q, which can be verified directly by using the Stolz–Cesàro theorem (Lemma A2, §A.5.1). The asymptotic order for $p(n)$ in (8.5) is exact.

The sum of m independent Logarithmic Series (p) variables has a MGF with the form (8.6) and Proposition 5 yields

$$p_m(n) \sim \frac{m}{(-\ln q)^m} (\ln n)^{m-1} n^{-1} p^n \qquad n \to \infty. \tag{8.8}$$

This mass function is highly intractable except when $m = 2$ as considered in §A.6.

Supplementary Materials

A complete copy of the paper with the Appendices below may be found at

www.smu.edu/Dedman/Academics/Departments/Statistics/Research/TechnicalReports

APPENDIX A: Contains proofs for asymptotic hazards, proofs using Feller conditions, examples with branch-point singularities, proofs for integer-valued distributions, and proofs with logarithmic singularities.

APPENDIX B: Contains proofs using Ikehara conditions and all derivations for compound distributions and first-passage distributions in SMPs.

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APPENDIX A: Asymptotic hazard rates, results using Feller conditions, examples with branch-point singularities, proofs for integer-valued distributions, and proofs with logarithmic singularities.

A.1 Asymptotic hazard rates

A.1.1 *Proof of Theorem 1*

First consider the case in which $b > 0$. For $0 < s < b$,

$$S(t) = P(e^{sX} > e^{st}) \leq e^{-st} \mathcal{M}(s)$$

by the Markov inequality. Thus,

$$H(t)/t \geq s - \ln \mathcal{M}(s)/t$$

and hence

$$\liminf_{t \to \infty} \{H(t)/t\} \geq s$$

for all $s < b$, so the liminf is at least b.

To show it cannot be more than b, suppose the contrary, that there is an $\varepsilon > 0$ such that

$$\liminf_{t \to \infty} \{H(t)/t\} \geq b + \varepsilon. \tag{8.9}$$

Then there exists a t_0 sufficiently large so that for all $t > t_0$

$$H(t)/t \geq b + \varepsilon/2 \qquad \text{or} \qquad -\ln\{S(t)\} = H(t) \geq (b + \varepsilon/2)t,$$

which implies that

$$S(t) \leq \exp\{-(b + \varepsilon/2)t\} \qquad t \geq t_0. \tag{8.10}$$

From this, we can now show that $\mathcal{M}(s)$ is convergent for $s < b + \varepsilon/2$ which contradicts the assumption of convergence bound b. To see this, use integration by parts and the triangle inequality for $t_0 < t_1$ to get

$$\left| \int_{t_0}^{t_1} e^{st} dS(t) \right| \leq s \int_{t_0}^{t_1} S(t) e^{st} dt + e^{st_0} S(t_0) + e^{st_1} S(t_1). \tag{8.11}$$

As $t_1 \to \infty$, the right side of (8.11) converges to a finite value for $s < b + \varepsilon/2$ by inequality (8.10); thus the left hand side is convergent and so also is $\mathcal{M}(s)$.

The $b = 0$ case follows from the proof given above. The first half of the proof is unnecessary since it is already known that $\liminf_{t \to \infty} \{H(t)/t\} \geq 0$. The second part presupposes (8.9) with $b = 0$ and reaches a contradiction to the fact that the convergence bound is 0 by using the same argument.

A.1.2 Extension of Theorem 1 to X with support $(-\infty, \infty)$

Let X^+ have the conditional distribution of X given $X > 0$. Let $H(t)$ be the cumulative hazard function for X with support on $(-\infty, \infty)$ and let $H_{X^+}(t)$ be the cumulative hazard for X^+. For $t > 0$, $S_{X^+}(t) = S(t)/S(0)$ so

$$H_{X^+}(t) = -\ln S_{X^+}(t) = -\ln S(t) + \ln S(0) = H(t) + \ln S(0).$$

Thus,

$$\liminf_{t \to \infty} \frac{H(t)}{t} = \liminf_{t \to \infty} \frac{H_{X^+}(t)}{t}. \tag{8.12}$$

Since the MGF of X^+ converges on $(-\infty, b)$ or $(-\infty, b]$, Theorem 1 applies to X^+ so the common liminf in (8.12) is b.

A.1.3 Proof of Theorem 2

The proof mimics that for the continuous setting of Theorem 1 so only some of the differences are outlined. First consider the case in which $b > 0$ and take $0 < s < b$. The survival function

$$S(n) = P(X \geq n) = \prod_{k=0}^{n-1}(1 - h_k) = \exp\left\{\sum_{k=0}^{n-1}\ln(1 - h_k)\right\}$$

satisfies a Markov inequality $S(n) \leq e^{-ns}\mathcal{M}(s)$ so that

$$-\frac{1}{n}\sum_{k=0}^{n-1}\ln(1 - h_k) \geq s - \frac{1}{n}\ln\mathcal{M}(s). \tag{8.13}$$

Taking $\liminf_{n \to \infty}$ of both sides, then the left side of (8.13) is $\geq s$ for all $s < b$, or $\geq b$.

Now suppose the $\liminf_{n \to \infty}$ of the left side of (8.13) is greater than $b + \varepsilon$ for some $\varepsilon > 0$. Then there is an n_0 such that $S(n) \leq \exp\{-n(b+\varepsilon/2)\}$ for all $n \geq n_0$. Thus $\mathcal{M}(s)$ must be convergent for $b < s < b + \varepsilon/2$, which is a contradiction.

The case in which $b = 0$ only requires the second portion of the above argument.

A.2 Proofs of basic results under Feller conditions

A.2.1 Proof of Proposition 1

Sufficiency of Feller conditions $\mathfrak{F}_{\mathcal{M}} \cap \mathfrak{F}_{UM}$ follows from Tauberian Theorem 4 of Feller (1971, XIII.5, p. 446) which is based on the Hardy-Littlewood-Karamata theorem in Theorem 2 (p. 445). The Laplace transform for $f_b(t)$ is

$$\mathcal{F}_b(s) = \mathcal{M}(b - s) \sim L(1/s)s^{-w} \qquad s \downarrow 0, \tag{8.14}$$

where $L(s) = g(b - 1/s)$ is slowly varying at ∞. Thus, if $f_b(t)$ is ultimately monotone as in condition $\mathfrak{F}_{UM}$, then it satisfies the conditions of Theorem 4 in Feller so $f_b(t) \sim g(b)t^{w-1}/\Gamma(w)$ and the order for $f(t)$ in (3.4) holds. Alternatively, if $f_b(t) \sim v(t)$ and $v(t)$ is ultimately monotone as in $\mathfrak{F}_{UM2}$, then $f_b(t)$ satisfies the conditions for the extended version of Theorem 4 given in Problem 16 of Feller (1971, XIII) and the same asymptotic order holds for $f_b(t)$.

The remainder of the arguments for the tail behaviour of $S(t)$ and $h(t)$ are the same as under Ikehara conditions.

A.2.2 *Example 2 (Excess life) under Feller conditions*

If X satisfies the conditions $\mathfrak{F}_{\mathcal{M}} \cap \mathfrak{F}_{UM}$ required for Proposition 1, then it is a simple exercise to show that E satisfies the conditions $\mathfrak{F}_{\mathcal{M}} \cap \mathfrak{F}_{UM2}$. For example, Proposition 1 applied to X gives $S(t) \sim f(t)/b$ and since $e^{bt}f(t)$ is ultimately monotone, $e^{bt}f_E(t) = e^{bt}S(t)/\mu \sim e^{bt}f(t)/(b\mu)$ which is an ultimately monotone function; thus $\mathfrak{F}_{UM2}$ holds for E and $\mathfrak{F}_{\mathcal{M}}$ also holds. Now Proposition 1 applies to E and gives $S_E(t) \sim f(t)/(b^2\mu)$.

A.2.3 *Extension of Proposition 1 to X with support $(-\infty, \infty)$*

Corollary A1. *Let X have an absolutely continuous distribution with support in $(-\infty, \infty)$ and moment generating function $\mathcal{M}(s)$ that converges on region $\{s \in C : a < \mathrm{Re}(s) < b\}$, for $-\infty \leq a \leq 0 < b < \infty$. If X satisfies Feller conditions $\mathfrak{F}_{\mathcal{M}} \cap \mathfrak{F}_{UM}$ or $\mathfrak{F}_{\mathcal{M}} \cap \mathfrak{F}_{UM2}$, then the limiting hazard rate is b and the density and survival functions have Gamma (w, b) tails as in (3.4) of Proposition 1.*

Proof. Let X^+ (X^-) have the conditional distribution of X given $X > 0$ $(X < 0)$. The corollary follows if Proposition 1 can be applied to the distribution of X^+, which means that properties for its density and MGF $\mathcal{M}^+(s)$, as required in Proposition 1, are inherited from the conditions of Corollary A1. Since the density for X^+ is $f^+(t) = f(t)/S(0)$, $f^+(t)$ inherits either property $\mathfrak{F}_{UM}$ or $\mathfrak{F}_{UM2}$ from $f(t)$. To show that $\mathcal{M}^+(s)$ inherits property $\mathfrak{F}_{\mathcal{M}}$ from $\mathcal{M}$, note that their relationship is given by

$$\mathcal{M}(s) = S(0)\mathcal{M}^+(s) + F(0)\mathcal{M}^-(s) \qquad s \in (a, b), \tag{8.15}$$

where $\mathcal{M}^+$ is convergent on $(-\infty, b)$ and $\mathcal{M}^-$, the MGF of X^-, is convergent on (a, ∞). Equating (8.15) with the expression for $\mathcal{M}$ in condition $\mathfrak{F}_{\mathcal{M}}$, then

$$\mathcal{M}^+(s) \sim S(0)^{-1}g(s)(b-s)^{-w} \qquad s \uparrow b$$

so that $\mathcal{M}^+$ satisfies $\mathfrak{F}_{\mathcal{M}}$ with $S(0)^{-1}g(s)$ left-continuous at $s = b$.

Applying Proposition 1 to the density of X^+, then

$$f^+(t) = \frac{f(t)}{S(0)} \sim \frac{g(b)}{S(0)}\frac{1}{\Gamma(w)}t^{w-1}e^{-bt},$$

which is the required order for $f(t)$. By following the remainder of the proof for Proposition 1, an asymptotic Gamma (w, b) tail emerges as well for the survival $S(t)$.

A.3 Convolution/mixture corollaries under Feller conditions

A.3.1 Corollary 2 using Feller conditions

Corollary A2. *Let X, Y, and Z be absolutely continuous and non-negative variables such that X is stronger than Y and Z; i.e. let $\mathcal{M}_X(s), \mathcal{M}_Y(s)$, and $\mathcal{M}_Z(s)$ converge on $\{\mathrm{Re}(s) < b\}, \{\mathrm{Re}(s) < b + \eta_Y\}$, and $\{\mathrm{Re}(s) < b + \eta_Z\}$ respectively for $b > 0$ and some values $\eta_Y > 0 < \eta_Z$.*

Feller conditions: *Assume X satisfies $\mathfrak{F}_{\mathcal{M}} \cap \mathfrak{F}_{ND(0,\infty)}$, with $\mathcal{M}_X(s) \sim g_X(s)(b-s)^{-w}$ as $s \uparrow b$ and $\mathfrak{F}_{ND(0,\infty)}$ given as:*

($\mathfrak{F}_{ND(0,\infty)}$) The tilted density $e^{bt} f_X(t)$ is non-decreasing for all $t > 0$.

Then $X + Y$ satisfies Feller conditions $\mathfrak{F}_{\mathcal{M}} \cap \mathfrak{F}_{ND(0,\infty)}$ and W satisfies $\mathfrak{F}_{\mathcal{M}} \cap \mathfrak{F}_{UM2}$. Thus, the asymptotic orders stated in (5.1) of Corollary 2 hold subject to the Feller conditions given above.

Proof. We apply Proposition 1 with Feller conditions $\mathfrak{F}_{\mathcal{M}} \cap \mathfrak{F}_{UM2}$ to the distribution of W. This distribution satisfies condition $\mathfrak{F}_{\mathcal{M}}$ since

$$\mathcal{M}_W(s) \sim p\mathcal{M}_Y(s)g_X(s)(b-s)^{-w} + q\mathcal{M}_Z(s) \sim p\mathcal{M}_Y(s)g_X(s)(b-s)^{-w}$$

as $s \uparrow b$. The latter equivalence follows since $\mathcal{M}_Z$ is convergent on $(-\infty, b + \eta_Z)$.

To show that condition $\mathfrak{F}_{UM2}$ holds, we show that $e^{bt} f_W(t) = f_{W,b}(t) \sim pf_{X+Y,b}(t)$ where $f_{X+Y,b}(t)$ is non-decreasing for all $t > 0$. The latter result holds based upon the following lemma.

Lemma A1. *Convolution $X + Y$ satisfies $\mathfrak{F}_{ND(0,\infty)}$ with tilting parameter b if X or Y satisfies $\mathfrak{F}_{ND(0,\infty)}$ with tilting parameter b.*

Proof. The b-tilted density for $X + Y$ is

$$f_{X+Y,b}(t) = \int_0^t f_{X,b}(t-u)f_{Y,b}(u)du = \int_0^t f_{X,b}(u)f_{Y,b}(t-u)du.$$

If X or Y satisfies $\mathfrak{F}_{ND(0,\infty)}$, then $X + Y$ satisfies $\mathfrak{F}_{ND(0,\infty)}$. This follows by using the same arguments as in Lemma B1 of §B.3.1. □

To see that $e^{bt} f_W(t) \sim pf_{X+Y,b}(t)$ so that W satisfies $\mathfrak{F}_{UM2}$, note that $\mathcal{M}_Z$ is convergent on $(-\infty, b + \eta_Z)$ so that its density $f_Z(t) = o(e^{-(b+\eta_Z/2)t})$ and

$$f_{W,b}(t) = pf_{X+Y,b}(t) + qf_{Z,b}(t) \sim pf_{X+Y,b}(t) \qquad t \to \infty.$$

A.3.2 Extension of Corollary A2 to $X, Y,$ and Z with support $(-\infty, \infty)$

Corollary A3. *Suppose absolutely continuous variables $X, Y,$ and Z have distributions on $(-\infty, \infty)$ with X stronger than Y and Z; i.e. $\mathcal{M}_X(s)$ converges on $(a, b) \ni 0$, and $\mathcal{M}_Y(s)$ and $\mathcal{M}_Z(s)$ converge on $[0, b + \eta_Y)$ and $[0, b + \eta_Z)$ respectively for some values $\eta_Y > 0 < \eta_Z$.*

Feller conditions: *Assume X satisfies $\mathfrak{F}_{\mathcal{M}} \cap \mathfrak{F}_{ND(-\infty, \infty)}$ (see below) so that $\mathcal{M}_X(s) \sim g_X(s)(b-s)^{-w}$ as $s \uparrow b$.*

($\mathfrak{F}_{ND(-\infty, \infty)}$) The tilted density $e^{bt} f_X(t)$ is non-decreasing for all $t \in (-\infty, \infty)$.

Then, the distribution of W has asymptotic hazard rate $b > 0$, with density and survival as in (5.1).

Proof. The distribution of W must satisfy conditions $\mathfrak{F}_{\mathcal{M}} \cap \mathfrak{F}_{UM2}$ of Corollary A1. Under the assumed conditions, $\mathcal{M}_W(s) \sim p\mathcal{M}_Y(s)g_X(s)(b-s)^{-w}$ as $s \uparrow b$ so $\mathfrak{F}_{\mathcal{M}}$ holds. To show $\mathfrak{F}_{UM2}$, we use the same arguments as used for Corollary A2 and Lemma A1 to show that $e^{bt} f_W(t) \sim p f_{X+Y,b}(t)$ where $f_{X+Y,b}(t)$ is non-decreasing for all $t \in (-\infty, \infty)$. From these arguments, we concluded that $X + Y$ satisfies $\mathfrak{F}_{ND(-\infty, \infty)}$ which allows W to satisfy $\mathfrak{F}_{UM2}$ and Corollary A1 to apply to W.

A.3.3 Corollary 4 using Feller conditions

Corollary A4. *Suppose absolutely continuous non-negative independent variables $X, Y,$ and Z have moment generating functions $\mathcal{M}_X(s), \mathcal{M}_Y(s),$ and $\mathcal{M}_Z(s)$ which share the common convergence region $\{s \in C : \mathrm{Re}(s) < b\}$.*

Feller conditions: *Let $X, Y,$ and Z all satisfy $\mathfrak{F}_{\mathcal{M}}$ of Proposition 1 with singularities at $b > 0$ of order w_X, w_Y, and w_Z respectively; thus $\mathcal{M}_X(s) \sim g_X(s)(b-s)^{-w_x}$, etc. Furthermore, suppose either X or Y satisfies $\mathfrak{F}_{ND(0, \infty)}$ and Z satisfies $\mathfrak{F}_{UND}$ if $p < 1$; i.e. $\mathfrak{F}_{UM}$ holds and $e^{bt} f_Z(t)$ is ultimately non-decreasing.*

Then, $X + Y$ satisfies $\mathfrak{F}_{\mathcal{M}} \cap \mathfrak{F}_{ND(0, \infty)}$ and W satisfies $\mathfrak{F}_{\mathcal{M}} \cap \mathfrak{F}_{UND}$ so that Proposition 1 applies. The asymptotic orders for $f_W(t)$ and $S_W(t)$ are as given in (5.2) and (5.3).

Proof. The proof that W satisfies the Feller conditions of Proposition 1 follows along the same lines as the proof of Corollary A2 with either X or Y satisfying $\mathfrak{F}_{ND(0, \infty)}$. In particular, the proof of condition $\mathfrak{F}_{UND}$, that $e^{bt} f_W(t)$ is ultimately non-decreasing, is exactly the same. Condition $\mathfrak{F}_{\mathcal{M}}$ holds with $\mathcal{M}_W(s) \sim g_W(b)(b-s)^{-w_*}$.

A.3.4 Corollary 5 using Feller conditions

Corollary A5. *(Convolution of i.i.d. variables). Consider* $W = X_1 + \cdots + X_m$ *where* $X_1, \ldots, X_m$ *are non-negatively-valued i.i.d. variables from an absolutely continuous distribution.*

Feller conditions: *If* X_1 *satisfies* $\mathfrak{F}_\mathcal{M} \cap \mathfrak{F}_{ND(0,\infty)}$ *with singularity* $b > 0$ *for* $\mathcal{M}_{X_1}$ *of order* w, *then* W *satisfies* $\mathfrak{F}_\mathcal{M} \cap \mathfrak{F}_{ND(0,\infty)} \subset \mathfrak{F}_\mathcal{M} \cap \mathfrak{F}_{UND}$ *and Proposition 1 applies to give (5.4) and the same conclusions as in Corollary 5.*

The proof uses the same argument as in Corollary 5.

A.4 Examples with branch-point singularities

We consider some distributions for which the existence theorems for asymptotic hazard rates do not apply.

Example A1. (Busy period for a random walk). Consider a continuous-time Markov chain on $\mathcal{S} = \{0, 1, \ldots\}$ which is a time-homogeneous birth-death process with transition rates from $i \to i+1$ and $i \to i-1$ as $\lambda > 0$ and $\mu > 0$ respectively. If the chain is positive recurrent, so $\lambda < \mu$, then the busy period for a random walk is the first-passage time from $1 \to 0$ and has MGF

$$\mathcal{M}(s) = \frac{2\mu}{\lambda + \mu - s} \frac{1}{1 + \sqrt{1 - 4\lambda\mu(\lambda + \mu - s)^{-2}}} \tag{8.17}$$

which converges on $\{s \in C : \mathrm{Re}(s) \leq b\}$ with $b = \lambda + \mu - 2\sqrt{\lambda\mu}$; see Butler (2007, eqn. 13.46). Aalen and Gjessing (2001) have shown the limiting hazard rate is b, the branch point of $\mathcal{M}(s)$.

Example A2. The Borel-Tanner (m, τ) mass function is

$$p(n) = \frac{m}{(n-m)!} n^{n-m-1} \tau^{n-m} e^{-\tau n} \qquad n \geq m \tag{8.18}$$

with $\tau \in (0, 1)$, and is the distribution for the number of customers X serviced in an $M/D/1$ queue before the queue reaches length 0; see Johnson and Kotz (1969, §10.6). The queue has a single server, starts with queue size m, and each customer has a fixed service time of T. Additional customers arrive according to a Poisson process with rate $\lambda > 0$ and $\tau = \lambda T$ is the stability parameter. The MGF and its convergence bound b may be determined from an implicit relationship for the MGF of X determined by Haight and Breuer (1960) when $\tau < 1$, so the queue is stable (positive recurrent). If $\mathcal{K}(s) = \ln \mathcal{M}(s)$, then Haight and Breuer (1960) give the relationship

$$\mathcal{K}(s) = m\tau \exp\{\mathcal{K}(s)/m\} + m(s - \tau). \tag{8.19}$$

The implicit relationship in (8.19) may be solved for $\mathcal{K}(s)$ to give

$$\mathcal{K}(s) = \ln \mathcal{M}(s) = -m\,\text{LambertW}(-\tau e^{s-\tau}) + m(s-\tau) \qquad s \in (-\infty, b]$$

where LambertW indicates the principal branch of the so-named function as discussed in Corless *et al.* (1996). On the real line, this principal branch is defined for $[-1/e, \infty)$. However, with the negative argument $-\tau e^{s-\tau} \in [-1/e, 0)$, the allowable range for s is $s \in (-\infty, b]$ for $b = \tau - \ln \tau - 1$. The value $-1/e$ is a branch point for LambertW and accordingly b is a branch point for $\mathcal{K}(s)$. The Stolz–Cesàro theorem (Lemma A2, §A.5.1) can be used to show the limiting hazard is $1 - e^{-b} = 1 - \tau e^{1-\tau}$. However, the existence of a limit for $\{h_n\}$ does not follow from theorems in the paper. Only Theorem 2 is applicable from which we conclude that

$$\liminf_{n \to \infty} \{-n^{-1} \sum_{k=0}^{n-1} \ln(1 - h_k)\} = b = \tau - \ln \tau - 1.$$

A.5 Proofs in integer time

Two preliminary results are needed in order to prove Proposition 2.

A.5.1 *Darboux's theorem and the Stolz–Cesàro lemma*

Theorem A1. *(Darboux). Suppose sequence $\{a_n\}$ has generating function $\mathcal{A}(z)$ which converges on the open disk of radius $c > 0$ and has the form*

$$\mathcal{A}(z) = g_0(z)(c - z)^{-w} + h_0(z) \tag{8.20}$$

with $w > 0$, $g_0(c) \neq 0$, and $g_0(z)$ and $h_0(z)$ are analytic on the closed disc $|z| \leq c$. Then

$$a_n \sim \tilde{a}_n = \frac{g_0(c)}{\Gamma(w)} n^{w-1} c^{-(n+w)} \qquad n \to \infty. \tag{8.21}$$

Proof. The term $\tilde{a}_n$ is the dominant term in the expansion given in Theorem 5.11 of Wilf (2006) adapted to a GF $\mathcal{A}(z)$ that converges on a disc of radius $c > 0$. To show this, rescale using a tilted mass function $d_n = c^n a_n$ so $\{d_n\}$ has GF

$$\mathcal{D}(z) = \mathcal{A}(cz) = g_0(cz)c^{-w}(1 - z)^{-w} + h_0(cz)$$

which converges on the unit disc. Using Theorem 5.11 of Wilf (2006) applied to $\{d_n\}$, $c^n a_n = d_n \sim g_0(c)c^{-w} n^{w-1}/\Gamma(w)$ so that (8.21) holds. $\square$

Lemma A2. *(Stolz–Cesàro). Let $\{a_n\}$ and $\{b_n\}$ be sequences that converge to zero, and assume that $\{b_n\}$ is strictly decreasing for large n. If*

$$\lim_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \gamma \qquad \text{(finite or infinite)},$$

then

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \gamma.$$

A.5.2 *Proof of Proposition 2*

We apply Theorem A1 with $a_n = p(n)$ and $\mathcal{A}(z) = g(z)(e^b - z)^{-w} + h(z)$ as its probability GF. The conditions in Proposition 2 ensure that $\mathcal{A}(z)$ satisfies the conditions of Theorem A1 through the mapping $z = e^s$. Thus, (8.21) provides the asymptotic order for $p(n)$ as indicated in (4.2) with $c = e^b$.

Since $\tilde{a}_{n+1}/\tilde{a}_n \to c^{-1} = e^{-b} < 1$ as $n \to \infty$, the tail sum $b_n = \sum_{k \geq n} \tilde{a}_k$ is convergent and strictly decreasing to 0. Applying Lemma A2 to this tail sum with $a_n = S(n)$, then $\gamma = 1$ and $S(n) \sim b_n$. Since $\{p(n) : n \geq 0\}$ cannot be 0 infinitely often as argued below, the limit of the inverse hazard $1/h_n = S(n)/p(n)$ can be found by considering

$$1/h_n \sim b_n/\tilde{a}_n = \sum_{k \geq n} (k/n)^{w-1} e^{-b(k-n)} = \sum_{m=0}^{\infty} (1 + m/n)^{w-1} e^{-bm}. \qquad (8.22)$$

The limit of the right summation, as $n \to \infty$, is given by the monotone convergence theorem. The sequence of summands in n is monotone decreasing (increasing) for $w > 1$ ($w < 1$) and constant for $w = 1$. Thus, in all cases, the limiting sum is $(1 - e^{-b})^{-1}$ and the limiting hazard rate in Proposition 2 holds.

The asymptotic order of $p(n)$ in Theorem A1 prevents $\{p(n) : n \geq 0\}$ from being 0 infinitely often. This is because if it were so, then $p(n)/\{n^{w-1} e^{-bn}\}$ could not converge to the positive number $g(e^b)e^{-bw}/\Gamma(w) > 0$. $\square$

A.5.3 *Extension of Proposition 2 to integer support on* $(-\infty, \infty)$

Corollary A6. *Let X have integer support and moment generating function $\mathcal{M}(s)$ that converges on $\{s \in C : a < \mathrm{Re}(s) < b\}$ with $-\infty \leq a \leq 0 < b < \infty$. Suppose b is a dominant singularity in that $\mathcal{M}(s)$ has the form*

$$\mathcal{M}(s) = g(e^s)(e^b - e^s)^{-w} + h(e^s)$$

where $w > 0$, $g(e^s)$ is analytic on $\{s \in C : \mathrm{Re}(s) \leq b\}$ with $g(e^b) \neq 0$, and $h(e^s)$ is analytic on $\{s \in C : a < \mathrm{Re}(s) \leq b\}$. Then the limiting hazard is $1 - e^{-b}$ and the mass and survival functions have asymptotic orders as given in (4.2).

Proof. The argument is the same as that used to prove Corollary B1 of §B.2.2 under conditions $\Im'_{\mathcal{M}} \cap \Im_{UND}$. $\square$

A.6 Logarithmic singularities and proofs for Propositions 4 and 5

The following theorem is a special case of a Tauberian theorem given in Narkiewiez (1983, thm. 3.10, p. 122).

Theorem A2. *Suppose $G(t)$ is a non-negative and non-decreasing function defined on $t \geq 0$ with Laplace transform $\mathcal{G}(s)$, which converges on $\{s \in C : \mathrm{Re}(s) > c\}$, and has the form*

$$\mathcal{G}(s) = \sum_{j=1}^{m} g_{0j}(s) \{-\ln(s-c)\}^j + h_0(s), \qquad (8.23)$$

where $g_{0m}(c) \neq 0$, and $\{g_{0j}(s)\}$ and $h_0(s)$ are analytic on $\{s \in C : \mathrm{Re}(s) \geq c\}$. Then,

$$G(t) \sim m g_{0m}(c) (\ln t)^{m-1} t^{-1} e^{ct} \qquad t \to \infty.$$

Proof of Proposition 4. The proof follows by using the same arguments that were used in Proposition 1 so only those arguments that differ are given. From (8.2), tilted density $f_{b+\varepsilon}(t)$ has Laplace transform

$$\mathcal{F}_{b+\varepsilon}(s) = \mathcal{M}(b + \varepsilon - s) = \sum_{j=1}^{m} g_j(b + \varepsilon - s) \{-\ln(s - \varepsilon)\}^j + h(b + \varepsilon - s),$$

which is convergent on $\{s \in C : \mathrm{Re}(s) > \varepsilon\}$ and which satisfies the conditions of Theorem A2 with $\mathcal{G}(s) = \mathcal{F}_{b+\varepsilon}(s)$. Therefore, $f_{b+\varepsilon}(t) \sim m g_m(b)(\ln t)^{m-1} t^{-1} e^{\varepsilon t}$ so the order for $f(t)$ in (8.1) holds.

Proof of the asymptotic order for the survival and hazard functions follows the approach used in Proposition 1 [see (8.28), (8.29), and (8.30)]. We need to show the equivalent of (8.30) or

$$\lim_{t \to \infty} \frac{1}{h(t)} = \lim_{t \to \infty} \frac{\int_t^\infty (\ln u)^{m-1} u^{-1} e^{-bu} du}{(\ln t)^{m-1} t^{-1} e^{-bt}}$$

$$= \lim_{t \to \infty} \int_0^\infty \left\{ 1 + \frac{\ln(1 + v/t)}{\ln t} \right\}^{m-1} \left(1 + \frac{v}{t} \right)^{-1} e^{-bv} dv, \qquad (8.24)$$

where substitution $v = u - t$ has been used. We can pass the limit through the integral by using the dominated convergence theorem to get $\int_0^\infty e^{-bv} dv = 1/b$. To justify using this theorem, note that $\ln(1 + v/t) < v/t$ so that the integrand in (8.24) is dominated by the integrable function $(1 + v)^m e^{-bv}$. Thus, the limiting hazard is b and $S(t) \sim f(t)/b$.

Confirm Examples 11 and 12.

We confirm (8.4) for the case $m = 2$. Convolve (8.3) with itself to get

$$f_{X_2}(t) = E_1(1)^{-2} e^{-t} \int_1^{t-1} u^{-1}(t-u)^{-1} du = E_1(1)^{-2} 2 \ln(t-1) t^{-1} e^{-t}$$

for $t > 2$ which is asymptotically equivalent to (8.4) when $m = 2$.

We confirm (8.8) for $m = 2$. The mass function in this setting is the convolution

$$p_2(n) = (-\ln q)^{-2} p^n \sum_{k=1}^{n} \frac{1}{k(n-k)} = (-\ln q)^{-2} p^n \frac{1}{n} \sum_{k=1}^{n} \left(\frac{1}{k} + \frac{1}{n-k} \right)$$

$$\sim (-\ln q)^{-2} p^n n^{-1} 2 \ln n,$$

which agrees with (8.8) for $m = 2$.

Proof of Proposition 5. The asymptotic order in (8.5) follows from the singularity analysis proposed in Flajolet and Odlyzko (1990). The tilted mass function $e^{bn} p(n)$ has probability GF that can be written in the form

$$\mathcal{P}(z) = \sum_{j=1}^{m} g_{j0}(z) \{-\ln(1-z)\}^j + h_0(z) \tag{8.25}$$

where $\{g_{j0}(z)\}$ and $h_0(z)$ are expressed in terms of $\{g_j(z)\}$ and $h(z)$ and are analytic on $\{z \in C : |z| \leq 1\}$. In particular, equality $g_{m0}(z) = g_m(ze^b)$ is used to get (8.5). Thus, $\mathcal{P}(z)$ is analytic on the closed unit disc $\{|z| \leq 1\}$ apart from a single logarithmic singularity on the boundary. Following the analysis in Flajolet and Odlyzko (1990), we determine the overall asymptotic order for the nth Taylor coefficient of $\mathcal{P}(z)$ as the dominant order from amongst the individual asymptotic orders of terms contributed by all of the $m + 1$ addends in (8.25). For addend $h_0(z)$, its convergence on $\{|z| \leq 1\}$ combined with compactness of the unit disc ensures the existence of an $\varepsilon > 0$ such that $h_0(z)$ converges on $\{|z| < 1 + \varepsilon\}$. Thus it contributes a term whose asymptotic order is at best $O\{(1 + \varepsilon)^{-n}\}$ and this is not dominant. For the other addends, Theorem 3A in Flajolet and Odlyzko (1990) can be applied accompanied by the comments in the last paragraph of page 231. Thus, the jth addend in (8.25) contributes a term of asymptotic order

$$g_{j0}(1) \frac{1}{n} (\ln n)^{j-1} \left\{ (-1) \binom{j}{1} \frac{d}{ds} \left. \frac{1}{\Gamma(-s)} \right|_{s=0} \right\} = g_{j0}(1) \frac{1}{n} (\ln n)^{j-1} j. \tag{8.26}$$

Among the expressions in (8.26), the mth term is dominant so their analysis gives

$$e^{bn} p(n) \sim g_{m0}(1) \frac{1}{n} (\ln n)^{m-1} m = g_m(e^b) \frac{1}{n} (\ln n)^{m-1} m$$

as in (8.5).

The remainder of the proof uses the same line of argument as used in the proof of Proposition 2. Following this line of argument, the inverse hazard has computational form much like in (8.24) with the limit computed as

$$1/h_n = \sum_{k=0}^{\infty} \left\{ 1 + \frac{\ln(1 + k/n)}{\ln n} \right\}^{m-1} (1 + k/n)^{-1} e^{-kb} \to (1 - e^{-b})^{-1} \qquad n \to \infty.$$

The limit is justified by the dominated convergence theorem. Since $\ln(1+k/n) < k/n$, the sequence of summands is bounded above by sequence $(1 + k)^m e^{-kb}$, which is summable in k.

APPENDIX B: PROOFS WITH IKEHARA CONDITIONS $\mathfrak{I}_\mathcal{M} \cap \mathfrak{I}_{UND}$, INCLUDING DERIVATIONS FOR CRAMÉR-LUNDBERG AND SPARRE ANDERSEN MODELS, COMPOUND DISTRIBUTIONS, AND SEMI-MARKOV PROCESSES

B.1 Proof of Proposition 1

The proof relies on the Ikehara-Wiener theorem when $w = 1$ and the Ikehara-Delange theorem when $0 < w \neq 1$.

B.1.1 *Ikehara-Wiener theorem*

A version of this theorem with slightly stronger assumptions than in Chandrasekharan (1968, p. 124), Doetsch (1950, p. 524), or Korevaar (2004, thm. 4.2, p. 124) is as follows.

Theorem B1. *(Ikehara-Wiener). Suppose $G(t) = 0$ for $t < 0$, is non-decreasing, right-continuous, and such that its Laplace transform $\mathcal{G}(s)$ exists for $\{s \in C : \mathrm{Re}(s) > 1\}$. If $\mathcal{G}(s)$ can be extended to a function that is analytic on the boundary $\{s \in C : \mathrm{Re}(s) = 1\}$, save from a simple pole at $s = 1$ with residue $\gamma > 0$, then*

$$G(t) \sim \gamma e^t \quad \text{as} \quad t \to \infty.$$

The restriction of this theorem to a function $G(t)$ whose transform $\mathcal{G}(s)$ is convergent on $\{s \in C : \mathrm{Re}(s) > 1\}$ is unnecessary. The convergence region may be $\{s \in C : \mathrm{Re}(s) > c\}$ with a simple dominant pole at $c > 0$. The theorem, however, does not allow for $c = 0$ as in Theorem 4 from Feller (1971, XIII.5). The corresponding conclusion when $c \neq 1$ is that the original function $G(t) \sim \gamma e^{ct}$ as $t \to \infty$. To show this, rescale $G(t)$ by defining $H(t) = G(t/c)$ with transform $\mathcal{H}(s) = c\mathcal{G}(cs)$ which has convergence region $\{\mathrm{Re}(s) > 1\}$ and a simple dominant pole at $s = 1$. Both transforms have the same residue at their respective poles since

$$\mathrm{Res}\{\mathcal{H}(s); s = 1\} = \lim_{s \to 1}\{(s - 1)c\mathcal{G}(cs)\} = \lim_{s \to c}\{(s - c)\mathcal{G}(s)\} = \gamma.$$

If $G(t)$ and $\mathcal{G}(s)$ satisfy the remaining conditions of Theorem B1, then so do $H(t)$ and $\mathcal{H}(s)$. Thus, applying Theorem B1 to $H(t)$ gives $G(t/c) = H(t) \sim \gamma e^t$ so that $G(t) \sim \gamma e^{ct}$ as required.

B.1.2 *Ikehara-Delange theorem*

The theorem below has been given in Narkiewiez (1983, thm. 3.9, p. 119).

Theorem B2. *(Ikehara-Delange). Suppose $G(t)$ is a non-negative and non-decreasing function defined on $t \geq 0$ with Laplace transform $\mathcal{G}(s)$ that converges on the half-plane*

$\{s \in C : \text{Re}(s) > c\}$. *If* $\mathcal{G}(s)$ *has the form*

$$\mathcal{G}(s) = g_0(s)(s-c)^{-w} + h_0(s) \qquad (8.27)$$

with $w > 0$, $g_0(c) \neq 0$, *and functions* $g_0(s)$ *and* $h_0(s)$ *are analytic on the closure* $\{s \in C : \text{Re}(s) \geq c\}$, *then*

$$G(t) \sim g_0(c)\,\Gamma(w)^{-1} t^{w-1} e^{ct} \qquad t \to \infty.$$

The Ikehara-Wiener theorem is essentially the special case of this theorem with $w = 1$ and with some alternative assumptions that are implied by the slightly stronger assumptions in Theorem B2. The Ikehara-Wiener theorem also makes some additional (unnecessary) assumptions which include that $G(t)$ is right-continuous, $c = 1$, and $g_0(c) > 0$; see Theorem B1 above.

When w is not integer-valued, the Laplace transform $\mathcal{G}(s)$ in (8.27) is specified in terms of a multi-function of the form $(s - c)^{-w}$. Principal branch values are assumed which are real-valued for $s > c$ and are based on using a branch cut along $[-\infty, c]$.

B.1.3 *Proof of Proposition* 1

Condition $\mathfrak{I}_{UND}$ is needed to use the Ikehara-Delange theorem with $G(t) = f_{b+\varepsilon}(t)$ non-decreasing. This function does not need to be non-decreasing on $(0, A)$, for some $A > 0$, because it can be replaced with the non-decreasing function $G(t) = f_{b+\varepsilon}(t)1_{\{t > A\}}$ whose Laplace transform has the same singularities as $f_{b+\varepsilon}(t)$ and whose form in (8.27) shares the same first term $g_0(s)(s-c)^{-w}$ but with a different $h_0(s)$ function (that is also analytic on the closure). Therefore, as a matter of convenience, the proof assumes $A = 0$ without any loss in generality.

Denoting the Laplace transforms of $f_{b+\varepsilon}(t)$ and $f(t)$ as $\mathcal{F}_{b+\varepsilon}(s)$ and $\mathcal{F}(s)$ respectively, then

$$\mathcal{F}_{b+\varepsilon}(s) = \mathcal{F}(s - b - \varepsilon) = \mathcal{M}(b + \varepsilon - s)$$

is analytic on $\text{Re}(s) > \varepsilon > 0$. By assumption from (3.5), $\mathcal{F}_{b+\varepsilon}(s)$ may be written as

$$\mathcal{F}_{b+\varepsilon}(s) = g(b + \varepsilon - s)(s - \varepsilon)^{-w} + h(b + \varepsilon - s)$$

where $g(b+\varepsilon-s)$ and $h(b+\varepsilon-s)$ are analytic on $\{s \in C : \text{Re}(s) \geq \varepsilon\}$. The Ikehara-Delange theorem applies to $f_{b+\varepsilon}(t)$ so

$$f_{b+\varepsilon}(t) \sim g(b)t^{w-1}e^{\varepsilon t}/\Gamma(w)$$

and the left side of (3.4) holds.

To compute the tail behaviour of $S(t)$, define $\tilde{S}$ and $\tilde{f}$ such that

$$\tilde{S}(t) = \int_t^\infty \tilde{f}(u)du = \int_t^\infty g(b)\Gamma(w)^{-1}u^{w-1}e^{-bu}du. \qquad (8.28)$$

The asymptotic order for $f(t)$ as $t \to \infty$ says that, for any $\varepsilon > 0$, $|f(u)/\tilde{f}(u) - 1| < \varepsilon$ for $u > T_0$. Therefore,

$$\left| \frac{S(t)}{\tilde{S}(t)} - 1 \right| \leq \tilde{S}(t)^{-1} \int_t^\infty \left| f(u) - \tilde{f}(u) \right| du = \tilde{S}(t)^{-1} \int_t^\infty \left| \frac{f(u)}{\tilde{f}(u)} - 1 \right| \tilde{f}(u) du < \varepsilon \quad (8.29)$$

for $t > T_0$. Thus, $S(t) \sim \tilde{S}(t)$ and therefore

$$\lim_{t \to \infty} h(t) = \lim_{t \to \infty} \frac{f(t)}{S(t)} = \lim_{t \to \infty} \frac{\tilde{f}(t)}{\tilde{S}(t)} = b, \quad (8.30)$$

the known asymptotic hazard rate for the Gamma (w, b) distribution. This proves the limiting hazard rate and also gives $S(t) \sim f(t)/b$.

B.2 Example 2 and extension of Proposition 1 to X with support $(-\infty, \infty)$

B.2.1 *Example* 2. (Excess life distribution). We show that E with the excess life distribution satisfies $\mathfrak{I}_\mathcal{M} \cap \mathfrak{I}_{UND}$. To show $\mathfrak{I}_\mathcal{M}$, note that

$$\mathcal{M}_E(s) = \frac{1 - \mathcal{M}(s)}{-s\mu} = \frac{1 - g(s)(b-s)^{-w} - h(s)}{-s\mu}.$$

Using the fact that

$$1 = \mathcal{M}(0) = g(0)b^{-w} + h(0),$$

then

$$\mathcal{M}_E(s) = \left[\{g(0)b^{-w} + h(0)\} - g(s)(b-s)^{-w} - h(s) \right] / (-s\mu)$$

$$= g_E(s)(b-s)^{-w} + h_E(s)$$

where

$$g_E(s) = \frac{g(0)b^{-w}(b-s)^w - g(s)}{-s\mu} \quad \text{and} \quad h_E(s) = \frac{h(0) - h(s)}{-s\mu}.$$

Since w is a positive integer, both $g_E(s)$ and $h_E(s)$ are analytic on $\{\text{Re}(s) \leq b\}$ with removable singularities at $s = 0$. Also $g_E(b) = g(b)/(b\mu) \neq 0$ so $\mathfrak{I}_\mathcal{M}$ holds.

To show $\mathfrak{I}_{UND}$ holds, note that

$$f_{E,b+\varepsilon}(t) = e^{(b+\varepsilon)t} f_E(t) = e^{(b+\varepsilon)t} S(t)/\mu \quad (8.31)$$

and, for any $\delta > 0$,

$$\ln \frac{f_{E,b+\varepsilon}(t+\delta)}{f_{E,b+\varepsilon}(t)} = \delta(b+\varepsilon) - \{H(t+\delta) - H(t)\}. \quad (8.32)$$

To get a lower bound on the latter term, note that, since $h(t) \to b$,

$$H(t+\delta) - H(t) = \int_t^{t+\delta} h(u)\, du \le (b+1)\delta \qquad t > T_0. \tag{8.33}$$

Thus,

$$\ln \frac{f_{E,b+\varepsilon}(t+\delta)}{f_{E,b+\varepsilon}(t)} \ge \delta\{(b+\varepsilon) - (b+1)\} = \delta(\varepsilon - 1) \qquad t > T_0.$$

Choosing $\varepsilon > 1$ makes $f_{E,b+\varepsilon}(t)$ ultimately non-decreasing for $t > T_0$.

B.2.2 *Extension of Proposition 1 to X with support $(-\infty, \infty)$*

Corollary B1. *Let X have an absolutely continuous distribution with support in $(-\infty, \infty)$ and moment generating function $\mathcal{M}(s)$ that converges on region $\{s \in C : a < \mathrm{Re}(s) < b\}$, for $-\infty \le a \le 0 < b < \infty$. If X satisfies Ikehara conditions $\mathfrak{I}'_{\mathcal{M}} \cap \mathfrak{I}_{UND}$ (see $\mathfrak{I}'_{\mathcal{M}}$ below), then the limiting hazard rate is b and the density and survival functions have Gamma (w, b) tails as in (3.4) of Proposition 1.*

$(\mathfrak{I}'_{\mathcal{M}})$ Singularity $b > 0$ is dominant in that $\mathcal{M}(s)$ may be expressed as

$$\mathcal{M}(s) = g(s)(b - s)^{-w} + h(s), \tag{8.34}$$

where $w > 0$, $g(s)$ is analytic on $\{s \in C : \mathrm{Re}(s) \le b\}$ with $g(b) \ne 0$, and $h(s)$ is analytic on $\{s \in C : a < \mathrm{Re}(s) \le b\}$.

Proof. Let X^+ (X^-) have the conditional distribution of X given $X > 0$ $(X < 0)$. The corollary follows if Proposition 1 can be applied to the distribution of X^+, which means that properties for its density and MGF $\mathcal{M}^+(s)$, as required in Proposition 1, are inherited from the conditions of Corollary B1. The only inherited property that isn't obvious is that $\mathcal{M}^+$ inherits property $\mathfrak{I}_{\mathcal{M}}$ from $\mathcal{M}$ when it satisfies $\mathfrak{I}'_{\mathcal{M}}$. The relationship of these two functions is given by

$$\mathcal{M}(s) = S(0)\mathcal{M}^+(s) + F(0)\mathcal{M}^-(s) \qquad \mathrm{Re}(s) \in (a, b), \tag{8.35}$$

where $\mathcal{M}^+$ is analytic on $\{\mathrm{Re}(s) < b\}$ and $\mathcal{M}^-$, the MGF of X^-, is analytic on $\{a < \mathrm{Re}(s)\}$. Equating (8.35) with the expression for $\mathcal{M}$ in condition $\mathfrak{I}'_{\mathcal{M}}$, then $\mathcal{M}^+(s) = g^+(s)(b-s)^{-w} + h_1^+(s)$ where

$$g^+(s) = S(0)^{-1}g(s) \quad \text{and} \quad h_1^+(s) = S(0)^{-1}\{h(s) - F(0)\mathcal{M}^-(s)\}.$$

Factor g^+ inherits its analyticity on $\{\mathrm{Re}(s) \le b\}$ directly from g. For h_1^+, the analytic assumption on h ensures that h_1^+ is analytic on $\{a < \mathrm{Re}(s) \le b\}$. Since $h_1^+(s)$ also agrees with $\mathcal{M}^+(s) - g^+(s)(b-s)^{-w}$ on this region and the latter difference is analytic

on $\{\text{Re}(s) < b\}$, then $h_1^+(s)$ has an analytic continuation onto $\{\text{Re}(s) \leq b\}$, denoted as $h^+(s)$, that satisfies the equality $\mathcal{M}^+(s) = g^+(s)(b - s)^{-w} + h^+(s)$; thus $\mathcal{M}^+$ satisfies $\mathfrak{I}_{\mathcal{M}}$ with the same dominant singularity as $\mathcal{M}$ within $\{\text{Re}(s) \leq b\}$.

Applying Proposition 1 to the density of X^+, then

$$\frac{f(t)}{S(0)} \sim \frac{g^+(b)}{\Gamma(w)} t^{w-1} e^{-bt} = \frac{g(b)}{S(0)\Gamma(w)} t^{w-1} e^{-bt},$$

which is the required order for $f(t)$. An asymptotic Gamma (w, b) tail emerges as well for survival $S(t)$.

B.3 Proofs for convolution/mixture corollaries

B.3.1 *Proof of Corollary* **2**. We show that the distribution of W satisfies conditions $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{UND}$ of Proposition 1. For $\mathfrak{I}_{\mathcal{M}}$, write the MGF of W as

$$\mathcal{M}_W(s) = \{pg_X(s)\mathcal{M}_Y(s)\}(b - s)^{-w} + \{ph_X(s)\mathcal{M}_Y(s) + q\mathcal{M}_Z(s)\}, \qquad (8.36)$$

which has the form (3.5) in condition $\mathfrak{I}_{\mathcal{M}}$.

To show that W satisfies $\mathfrak{I}_{UND}$, we first determine that $X + Y$ satisfies $\mathfrak{I}_{ND(0,\infty)}$ if X or Y satisfies $\mathfrak{I}_{ND(0,\infty)}$.

Lemma B1. *Convolution $X + Y$ satisfies $\mathfrak{I}_{ND(0,\infty)}$ with tilting parameter $b^+ = b + \varepsilon$ if X or Y satisfies $\mathfrak{I}_{ND(0,\infty)}$ with tilting parameter b^+.*

Proof. Convolution $X + Y$ has tilted density

$$f_{X+Y,b^+}(t) = e^{b^+ t} f_{X+Y}(t) = \int_0^t e^{b^+(t-u)} f_X(t-u) e^{b^+ u} f_Y(u) du$$

$$= \int_0^t f_{X,b^+}(t-u) f_{Y,b^+}(u) du = \int_0^t f_{X,b^+}(u) f_{Y,b^+}(t-u) du. \qquad (8.37)$$

If X satisfies $\mathfrak{I}_{ND(0,\infty)}$ with tilting parameter $b^+ = b + \varepsilon$, then we work with the leftmost convolution in (8.37). With $t > 0$ and $\delta > 0$, the convolution gives the difference

$$f_{X+Y,b^+}(t + \delta) - f_{X+Y,b^+}(t) = \int_0^{t+\delta} f_{X,b^+}(t + \delta - u) f_{Y,b^+}(u) du - \int_0^t f_{X,b^+}(t-u) f_{Y,b^+}(u) du$$

$$= \int_t^{t+\delta} f_{X,b^+}(t + \delta - u) f_{Y,b^+}(u) du + \int_0^t \{f_{X,b^+}(t + \delta - u) - f_{X,b^+}(t-u)\} f_{Y,b^+}(u) du.$$

$$(8.38)$$

The first term in (8.38) is non-negative, and, since $f_{X,b^+}(t)$ satisfies $\mathfrak{I}_{ND(0,\infty)}$, the latter term is also non-negative so that $X + Y$ satisfies $\mathfrak{I}_{ND(0,\infty)}$. If instead, f_{Y,b^+} satisfies

$\mathfrak{I}_{ND(0,\infty)}$, then the same argument holds with the roles of X and Y interchanged and using the rightmost convolution in (8.37).

For the setting in which $p = 1$, the distribution of $W = X + Y$ now satisfies all conditions of Proposition 1. For the case in which $p < 1$, suppose the $(b + \varepsilon_1)$-tilted density for Z is ultimately non-decreasing and $\varepsilon_2 = \max(\varepsilon, \varepsilon_1)$. Then the $(b + \varepsilon_2)$-tilted density for W is ultimately non-decreasing since both terms of

$$pe^{(b+\varepsilon_2)t} f_{X+Y}(t) + qe^{(b+\varepsilon_2)t} f_Z(t) \tag{8.39}$$

are ultimately non-decreasing. The $(b+\varepsilon_2)$-tilted density for W now satisfies the remaining condition $\mathfrak{I}_{UND}$ of Proposition 1. $\square$

B.3.2 *Extension of Corollary 2 to* X, Y, and Z *with support* $(-\infty, \infty)$

Corollary B2. *Suppose absolutely continuous variables X, Y, and Z have distributions on $(-\infty, \infty)$ with X stronger than Y and Z; i.e. $\mathcal{M}_X(s)$ converges on $(a, b) \ni 0$, and $\mathcal{M}_Y(s)$ and $\mathcal{M}_Z(s)$ converge on $[0, b + \eta_Y)$ and $[0, b + \eta_Z)$ respectively for some values $\eta_Y > 0 < \eta_Z$.*

Ikehara conditions: *Assume X satisfies $\mathfrak{I}'_{\mathcal{M}}$ of Corollary B1, either X or Y (or both) satisfies $\mathfrak{I}_{ND(-\infty,\infty)}$ with $\mathfrak{I}_{ND(-\infty,\infty)}$ given below, Z satisfies $\mathfrak{I}_{UND}$, and $\mathcal{M}_Y(s)$ also converges on the range $s \in (-\infty, 0)$.*

($\mathfrak{I}_{ND(-\infty,\infty)}$) X satisfies $\mathfrak{I}_{ND(-\infty,\infty)}$ if there exists an $\varepsilon > 0$ for which $e^{(b+\varepsilon)t} f_X(t)$ is non-decreasing for all $t \in (-\infty, \infty)$.

Then, the distribution of W has asymptotic hazard rate $b > 0$, with density and survival as in (5.1).

Proof. We need to show that the distribution of W satisfies conditions $\mathfrak{I}'_{\mathcal{M}} \cap \mathfrak{I}_{UND}$ of Corollary B1. Under the assumed conditions, $\mathcal{M}_W$ is given in (8.36) and has the form (8.34) so condition $\mathfrak{I}'_{\mathcal{M}}$ holds with $a = 0$. To show $\mathfrak{I}_{UND}$, we use the same arguments as used in §B.3.1 for proving Corollary 2 and Lemma B1. These arguments imply that $X + Y$ satisfies $\mathfrak{I}_{ND(-\infty,\infty)}$ with tilting parameter $b + \varepsilon$ when either X or Y satisfies $\mathfrak{I}_{ND(-\infty,\infty)}$ with tilting parameter $b + \varepsilon$. Since we assume that $e^{(b+\varepsilon_3)t} f_Z(t)$ is ultimately non-decreasing, then $e^{(b+\varepsilon_4)t} f_W(t)$ is ultimately non-decreasing with $\varepsilon_4 = \max(\varepsilon, \varepsilon_3)$. Corollary B1 applies to give the results.

B.3.3 *Proof of Corollary* 4. The conclusions of the corollary hold if W satisfies conditions $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{UND}$ of Proposition 1. Condition $\mathfrak{I}_{\mathcal{M}}$ follows from the fact that X, Y, and Z satisfy $\mathfrak{I}_{\mathcal{M}}$ with w_X, w_Y, and w_Z as positive integers. Write $\mathcal{M}_W(s)$ in the form $g_W(s)(b-s)^{-w_*} + h_W(s)$. If w_X, w_Y, and w_Z are positive integers, then $g_W(s)$ is analytic

on $\{\mathrm{Re}(s) \leq b\}$; otherwise b may be a branch point for g_W when $(b-s)^{-w_*}$ is factored out to determine $g_W(s)$.

To show $\mathfrak{I}_{UND}$, we use Lemma B1 of §B.3.1 to conclude that $X+Y$ satisfies $\mathfrak{I}_{ND(0,\infty)}$ with tilting parameter $b+\varepsilon$ if X or Y satisfies $\mathfrak{I}_{ND(0,\infty)}$ with the same tilting parameter. Since by assumption, Z satisfies $\mathfrak{I}_{UND}$ for some $\varepsilon_1 > 0$, then the mixture density for W must satisfy $\mathfrak{I}_{UND}$ for $\varepsilon_2 = \max\{\varepsilon, \varepsilon_1\}$ so $\mathfrak{I}_{UND}$ holds for W.

B.3.4 *Proof of Example* 5. First, based on Example 2, the conditions on $\mathcal{M}_X$ ensure that E satisfies $\mathfrak{I}_\mathcal{M}$ (see the proof in §B.2.1) and that singularity b for $\mathcal{M}_E$ is a w-pole.

We now show that E satisfies $\mathfrak{I}_{ND(0,\infty)}$. Since X satisfies $\mathfrak{I}_{ND(0,\infty)}$, there is an $\varepsilon > 0$ such that

$$1 \leq \frac{f_{X,b+\varepsilon}(t+\delta)}{f_{X,b+\varepsilon}(t)} = e^{\delta(b+\varepsilon)}\frac{f_X(t+\delta)}{f_X(t)} \qquad t > 0; \delta > 0.$$

Thus, $e^{\delta(b+\varepsilon)}f_X(t+\delta) \geq f_X(t)$ and

$$e^{\delta(b+\varepsilon)}S_X(t+\delta) = e^{\delta(b+\varepsilon)}\int_t^\infty f_X(s+\delta)ds \geq \int_t^\infty f_X(s)ds = S_X(t).$$

The implication for $f_{E,b+\varepsilon}$ is

$$\frac{f_{E,b+\varepsilon}(t+\delta)}{f_{E,b+\varepsilon}(t)} = e^{\delta(b+\varepsilon)}\frac{S_X(t+\delta)}{S_X(t)} \geq 1 \qquad t > 0; \delta > 0,$$

so that E satisfies $\mathfrak{I}_{ND(0,\infty)}$ with the same tilting parameter $b + \varepsilon$.

Putting the two results together, then E satisfies $\mathfrak{I}_\mathcal{M} \cap \mathfrak{I}_{ND(0,\infty)}$ and the conditions for Corollary 5 are met.

B.4 Derivations for compound distributions

B.4.1 *Cramér-Lundberg – Proof of Theorem* 3

We show that R^+ satisfies $\mathfrak{I}_\mathcal{M} \cap \mathfrak{I}_{ND(0,\infty)}$. The MGF for R^+ is

$$\mathcal{M}_{R^+}(s) = \frac{\mathcal{M}_R(s) - (1-\rho)}{\rho} = \frac{(1-\rho)\mathcal{M}_E(s)}{1-\rho\mathcal{M}_E(s)} = g_{R^+}(s)(b-s)^{-1}, \qquad (8.40)$$

where

$$g_{R^+}(s) = \frac{(1-\rho)\mathcal{M}_E(s)}{\{1-\rho\mathcal{M}_E(s)\}/(b-s)}$$

is analytic on $\{\mathrm{Re}(s) \leq b\}$. This follows since the apparent singularity of g_{R^+} at b is removable and since $1 - \rho\mathcal{M}_E(s)$ cannot have a zero in an open neighbourhood of $\{\mathrm{Re}(s) \leq b\}$ apart from at $s = b$. To show the latter result, note that $|\mathcal{M}_E(b + iy)| < |\mathcal{M}_E(b)| = 1/\rho$ for all $y \neq 0$; see Daniels (1954, p. 632) for a proof. Thus, there is a

sufficiently small neighbourhood N_y of $b + iy$ on which $|\mathcal{M}_E(s)| < 1/\rho$ for $s \in N_y$. Consequently, $1 - \rho \mathcal{M}_E(s)$ can have no zeros on the open cover $\cup_{y \neq 0} N_y$ for $\{b + iy : y \neq 0\}$. On $\{\mathrm{Re}(s) < b\}$, $|\mathcal{M}_E(x + iy)| \leq \mathcal{M}_E(x) < \mathcal{M}_E(b) = 1/\rho$ when $x < b$. Thus, $1 - \rho \mathcal{M}_E(s)$ cannot be zero on the open cover $\{\mathrm{Re}(s) < b\} \cup_{y \neq 0} N_y$ for $\{\mathrm{Re}(s) \leq b\} \backslash \{b\}$. The singularity for g_{R^+} at b is removable and the version of g_{R^+} that is analytic on $\{\mathrm{Re}(s) \leq b\}$ uses l'Hôpital's rule to define

$$g_{R^+}(b) = \frac{(1-\rho)\mathcal{M}_E(b)}{\rho \mathcal{M}'_E(b)} = \frac{1-\rho}{\rho \left[d\{\ln \mathcal{M}_E(s)\}/ds|_{s=b} \right]}. \tag{8.41}$$

Expression (8.41) is positive since the cumulant generating function $\ln \mathcal{M}_E(s)$ of E is strictly increasing at $b < c$. Thus, $g_{R^+}(b) > 0$ and R^+ satisfies $\mathfrak{I}_{\mathcal{M}}$.

To show that R^+ satisfies $\mathfrak{I}_{ND(0,\infty)}$, we first determine a mixture expansion for the density of R^+ using

$$\mathcal{M}_{R^+}(s) = \sum_{k=0}^{\infty} (1-\rho)\rho^k \mathcal{M}_E^{k+1}(s).$$

Doetsch (1974, thm. 30.1) allows for a formal proof that the associated density has mixture form

$$f_{R^+}(t) = \sum_{k=0}^{\infty} (1-\rho)\rho^k f_{W_{k+1}}(t) \qquad \text{a.e. } t > 0, \tag{8.42}$$

where $W_{k+1} = E_1 + \cdots + E_{k+1}$ is a sum of i.i.d. excess life variables and (8.42) holds for a.e. $t > 0$. The right-hand side of (8.42) is actually a continuous version of density $f_{R^+}(t)$ for $t \geq 0$ as we now show. Since $f_E(t) \leq 1/\mu$, it is easy to show that $f_{W_{k+1}}(t) \leq 1/\mu$ for all $t \geq 0$ and all k. Thus,

$$\sum_{k=0}^{\infty} (1-\rho)\rho^k f_{W_{k+1}}(t) \leq \sum_{k=0}^{\infty} (1-\rho)\rho^k/\mu = 1/\mu$$

so (8.42) converges uniformly on $[0,\infty)$ by the Weierstrass M-test (Apostol, 1957, thm. 13.7). Since $f_E(t)$ is continuous on $[0,\infty)$, then $f_{W_{k+1}}(t)$ is also continuous on $[0,\infty)$ for all k; thus, the version of $f_{R^+}(t)$ given in (8.42) is continuous on $[0,\infty)$ (Apostol, 1957, thm. 13.8).

We now show that this continuous version of $f_{R^+}(t)$ satisfies $\mathfrak{I}_{ND(0,\infty)}$. Using the assumptions of Theorem 3 along with the results in Example 5, we conclude that E satisfies $\mathfrak{I}_{ND(0,\infty)}$; hence, each W_{k+1} also satisfies $\mathfrak{I}_{ND(0,\infty)}$. In fact, in showing that W_{k+1} satisfies $\mathfrak{I}_{ND(0,\infty)}$ with Lemma B1, the tilting parameter required for $\mathfrak{I}_{ND(0,\infty)}$ to hold does not depend upon k and is the same tilting parameter as required for $W_1 = E_1$. Thus, there exists an $\varepsilon > 0$ such that $e^{(c+\varepsilon)t} f_{W_{k+1}}(t)$ is non-decreasing for all $t > 0$ and $k \geq 0$. Using this and the expansion in (8.42), we conclude that the continuous version of $e^{(c+\varepsilon)t} f_{R^+}(t)$ given by (8.42) is non-decreasing for all $t > 0$ and satisfies $\mathfrak{I}_{ND(0,\infty)}$.

Applying Proposition 1 to R^+, then $f_{R^+}(t) \sim g_{R^+}(b)e^{-bt}$ where $g_{R^+}(b)$ is given in (8.41). We can rewrite this expression in terms of $\mathcal{M}_X$ by differentiating $\ln \mathcal{M}_E(s) = \ln\{1 - \mathcal{M}_X(s)\} - \ln(-\mu s)$ at $s = b$ to get

$$\left. \frac{d \ln \mathcal{M}_E(s)}{ds} \right|_{s=b} = -\frac{\mathcal{M}'_X(b)}{1 - \mathcal{M}_X(b)} - \frac{1}{b}. \tag{8.43}$$

Now, using the fact that b is a root of $1 - \rho \mathcal{M}_E(s) = 1 - \rho\{1 - \mathcal{M}_X(s)\}/(-\mu s)$, we can rewrite the right side of (8.43) as $b^{-1}\{\lambda \mathcal{M}'_X(b)/\sigma - 1\}$ so that (8.41) becomes

$$g_{R^+}(b) = \frac{(1-\rho)b}{\rho\{\lambda \mathcal{M}'_X(b)/\sigma - 1\}}.$$

This gives

$$f_{R^+}(t) \sim g_{R^+}(b)e^{-bt} = \frac{(1-\rho)b}{\rho\{\lambda \mathcal{M}'_X(b)/\sigma - 1\}}e^{-bt}$$

and $S_{R^+}(t) \sim f_{R^+}(t)/b$. Since also $S_R(t) = \rho S_{R^+}(t)$, then

$$S_R(t) \sim \frac{(1-\rho)}{\lambda \mathcal{M}'_X(b)/\sigma - 1}e^{-bt}$$

which agrees with the traditional Cramér-Lundberg approximation; see Asmussen (2000, III.5 thm. 5.3).

B.4.2 *Sparre Andersen – Proof of Theorem* 4

R^+ *satisfies condition* $\mathfrak{I}_{\mathcal{M}}$. The MGF $\mathcal{M}_R$ is determined through Wiener-Hopf factorization, as described in Feller (1971, XVIII, §§3-5), Prabhu (1980, ch. 2, §1), and Embrechts and Veraverbeke (1982, §2). When recast in terms of the MGF of $Y = X - \sigma T$ rather than its characteristic function, it states that

$$1 - \mathcal{M}_Y(s) = \{1 - \Psi^+(s)\}\{1 - \Psi^-(s)\} \qquad -a/\sigma < \mathrm{Re}(s) < c.$$

The factorization is unique among functions Ψ^+ and Ψ^- for which Ψ^+ is analytic on $\{\mathrm{Re}(s) < 0\}$ and Ψ^- is analytic on $\{\mathrm{Re}(s) > 0\}$. Functions Ψ^+ and Ψ^- are MGFs

$$\Psi^+(s) = \int_{0+}^{\infty} e^{st}dF^+(t) \qquad \Psi^-(s) = \int_{-\infty}^{0^+} e^{st}dF^-(t)$$

for the defective right and left Wiener-Hopf distributions, $F^+(t)$ and $F^-(t)$, whose support is on $(0,\infty)$ and $(-\infty,0]$ respectively. Negative drift of the random walk $\{S_n\}$ ensures that

$$B = \sum_{n=1}^{\infty} P(S_n > 0)/n < \infty, \tag{8.44}$$

$1 - \Psi^+(0) = e^{-B} < 1$, and $\Psi^-(0) = 1$. Under such conditions, the MGF of R is

$$\mathcal{M}_R(s) = e^{-B}/\{1 - \Psi^+(s)\} \qquad (8.45)$$

and is analytic on at least $\mathrm{Re}(s) < 0$. We now show that $\mathcal{M}_R(s)$ is analytic on $\{\mathrm{Re}(s) \le b\}$ apart from a simple dominant pole at $b > 0$.

First note that there are two real zeros of $1 - \mathcal{M}_Y(s)$ within $s \in (-a/\sigma, c)$ which are 0 and $b > 0$. Since $\mathcal{M}'_Y(0) < 0 < \mathcal{M}'_Y(b)$, both zeros are simple zeros. Since 0 is a zero of $1 - \Psi^-(s)$ and $1 - \Psi^-(s)$ is strictly increasing in s, the value b cannot also be its zero, hence $0 < 1 - \Psi^-(b) < \infty$. Thus, b must be a simple zero of $1 - \Psi^+(s)$ and hence it is a simple pole for $\mathcal{M}_R(s)$. Now, if there is any other singularity for $\mathcal{M}_R(s)$ within an open neighbourhood of $\{0 \le \mathrm{Re}(s) \le b\}$, then it has to be a zero of $1 - \Psi^+(s)$ and hence also a zero of $1 - \mathcal{M}_Y(s)$. We now show no such zeros can be found for $1 - \mathcal{M}_Y(s)$ within such an open neighbourhood apart from the zeros at 0 and b. First, along the lines $\{\mathrm{Re}(s) = b\}$ and $\{\mathrm{Re}(s) = 0\}$, there can be no zeros for $1 - \mathcal{M}_Y(s)$ apart from b and 0 since

$$|\mathcal{M}_Y(b + iy)| < \mathcal{M}_Y(b) = 1 \qquad y \ne 0$$
$$|\mathcal{M}_Y(iy)| < \mathcal{M}_Y(0) = 1 \qquad y \ne 0;$$

see Daniels (1954, p. 632). Also, $\Psi^+(0) < 1$ so 0 cannot be a zero of $1 - \Psi^+(s)$. As argued in the Cramér-Lundberg case, there exists an open covering of $\{b + iy : y \ne 0\} \cup \{iy : y \in (-\infty, \infty)\}$ on which $1 - \Psi^+(s)$ has no zeros. Also, there are no zeros within the region $\{0 < \mathrm{Re}(s) < b\}$ for the same reason:

$$|\mathcal{M}_Y(x + iy)| \le \mathcal{M}_Y(x) < \mathcal{M}_Y(b) = 1 \qquad x \in (0, b); y \in (-\infty, \infty).$$

Thus, apart from a simple pole at $b > 0$, $\mathcal{M}_R(s)$ is analytic over $\{0 \le \mathrm{Re}(s) \le b\}$.

The MGF of R^+ is determined by taking a geometric expansion of the MGF of R in (8.45) which is

$$\mathcal{M}_R(s) = e^{-B} \sum_{j=0}^{\infty} (\Psi^+)^j(s) = e^{-B} + (1 - e^{-B}) \sum_{k=0}^{\infty} e^{-B}(1 - e^{-B})^k \left\{ \frac{\Psi^+(s)}{\Psi^+(0)} \right\}^{k+1}, \quad (8.46)$$

where $\Psi^+(0) = 1 - e^{-B}$. The MGF for R^+ is the rightmost summation which is

$$\mathcal{M}_{R^+}(s) = \frac{e^{-B}\mathcal{M}_{L^+}(s)}{1 - (1 - e^{-B})\mathcal{M}_{L^+}(s)} = \mathcal{M}_R(s)\frac{\Psi^+(s)}{\Psi^+(0)}, \qquad (8.47)$$

where $\Psi^+(s)/\Psi^+(0) = \mathcal{M}_{L^+}(s)$ is the MGF of the ascending ladder distribution or the normalised right Wiener-Hopf distribution. The rightmost identity reveals that $\mathcal{M}_{R^+}$ and $\mathcal{M}_R$ have the same analyticity on $\{\mathrm{Re}(s) \le b\}\backslash\{b\}$ since $\Psi^+(s)$ is analytic on $\{\mathrm{Re}(s) \le b\}$. Thus R^+ satisfies $\mathfrak{I}_{\mathcal{M}}$.

R^+ *satisfies condition* $\mathfrak{I}_{UND}$. The geometric expansion in (8.46) demonstrates that R has a compound geometric sum characterization $R = \sum_{i=0}^{G} L_i^+$, where $\{L_i^+ : i \geq 1\}$ are i.i.d. with the ascending ladder distribution of the random walk $\{S_n\}$, G has a Geometric (e^{-B}) distribution on $\{0, 1, \ldots\}$, and L_0^+ places point mass e^{-B} at 0; see also Embrechts and Veraverbeke (1982, eqn. 10). Thus, the proof that R^+ satisfies $\mathfrak{I}_{UND}$ can follow the same derivation as used for Cramér-Lundberg. Doetsch (1974, thm. 30.1) allows for a formal proof that the associated density has mixture form

$$f_{R^+}(t) = \sum_{k=0}^{\infty} e^{-B}(1 - e^{-B})^k f_{L_{(k+1)}^+}(t) \qquad \text{a.e. } t > 0, \tag{8.48}$$

where $L_{(k)}^+ = L_1^+ + \cdots + L_k^+$ is a k-fold convolution. We use the right-hand side of (8.48) as the version of $f_{R^+}(t)$ that we shall show satisfies $\mathfrak{I}_{ND(0,\infty)}$. This condition is satisfied if we show $e^{(c+\varepsilon)t} f_{R^+}(t)$, as determined from (8.48), satisfies $\mathfrak{I}_{ND(0,\infty)}$ for some $\varepsilon > 0$. We show this holds if $e^{(c+\varepsilon)t} f_{L_1^+}(t)$ satisfies $\mathfrak{I}_{ND(0,\infty)}$. We show the latter result in two steps. (a) First we show that the tilted ladder density $e^{(c+\varepsilon)t} f_{L_1^+}(t)$ satisfies $\mathfrak{I}_{ND(0,\infty)}$ if $e^{(c+\varepsilon)t} f_Y(t)$ satisfies $\mathfrak{I}_{ND(0,\infty)}$. (b) Secondly, we show that $e^{(c+\varepsilon)t} f_Y(t)$ satisfies $\mathfrak{I}_{ND(0,\infty)}$ if the claim amount X satisfies $\mathfrak{I}_{ND(0,\infty)}$ as assumed in Theorem 4.

(a) Let $N = \inf\{n > 0 : S_n > 0\}$ be the ladder epoch for the ascending latter height with $N = \infty$ if $\sup_{n \geq 1} S_n \leq 0$. With $S_0 = 0$, if $N = n$ then the last step amount $S_n - S_{n-1} = Y_n$ is a positive increment that takes a negative S_{n-1} to a positive S_n. Thus, if

$$G_{n-1}(t) = \begin{cases} P(S_1 \leq t) & \text{if } t < 0, \ n = 2 \\ P(S_1 < 0, \ldots, S_{n-2} < 0, S_{n-1} \leq t) & \text{if } t < 0, \ n \geq 3 \end{cases},$$

then the joint density/mass function of L_1^+ and N given $N < \infty$ is

$$f_{L_1^+,N}(t, n) = (1 - e^{-B})^{-1} \begin{cases} f_Y(t) & \text{if } t > 0, \ n = 1 \\ \int_{-\infty}^{0} f_Y(t-u) dG_{n-1}(u) & \text{if } t > 0, \ n \geq 2 \end{cases}. \tag{8.49}$$

For example, if $n \geq 4$ then

$$G_{n-1}(u) = \int_{-\infty}^{0} f_{Y_1}(s_1) ds_1 \cdots \int_{-\infty}^{0} f_{Y_{n-2}}(s_{n-2} - s_{n-3}) F_{Y_{n-1}}(u - s_{n-2}) ds_{n-2}$$

can be derived through the transformation $(Y_1, \ldots, Y_{n-1}) \to (S_1, \ldots, S_{n-1})$. The marginal density of L_1^+ given $N < \infty$ is

$$f_{L_1^+}(t) = \sum_{n=1}^{\infty} f_{L_1^+,N}(t, n) = (1 - e^{-B})^{-1} \left\{ f_Y(t) + \sum_{n=2}^{\infty} \int_{-\infty}^{0} f_Y(t-u) dG_{n-1}(u) \right\}. \tag{8.50}$$

Tilting both sides of (8.50), then

$$f_{L_1^+,c+\varepsilon}(t) = (1-e^{-B})^{-1}\left\{f_{Y,c+\varepsilon}(t) + \sum_{n=2}^{\infty}\int_{-\infty}^{0}f_{Y,c+\varepsilon}(t-u)e^{(c+\varepsilon)u}dG_{n-1}(u)\right\}. \quad (8.51)$$

Take $\delta \geq 0$ and assume that $f_{Y,c+\varepsilon}(t)$ is non-decreasing for all $t > 0$ as a result of f_Y satisfying $\mathfrak{I}_{ND(0,\infty)}$. Then,

$$f_{Y,c+\varepsilon}(t+\delta) \geq f_{Y,c+\varepsilon}(t)$$

for all $t > 0$. Using (8.51), this inequality implies

$$f_{L_1^+,c+\varepsilon}(t+\delta) \geq f_{L_1^+,c+\varepsilon}(t)$$

for all $t > 0$ hence $f_{L_1^+}$ satisfies $\mathfrak{I}_{ND(0,\infty)}$.

(b) We first write $f_Y(t)$ as a convolution in f_X and $f_{-\sigma T}$ so that

$$f_Y(t) = \int_{-\infty}^{0} f_X(t-u)f_{-\sigma T}(u)du \qquad t > 0. \quad (8.52)$$

Thus,

$$f_{Y,c+\varepsilon}(t) = \int_{-\infty}^{0} f_{X,c+\varepsilon}(t-u)f_{-\sigma T,c+\varepsilon}(u)du \qquad t > 0. \quad (8.53)$$

Following the same proof as in part (a), then $f_{Y,c+\varepsilon}$ is non-decreasing for all $t > 0$ (f_Y satisfies $\mathfrak{I}_{ND(0,\infty)}$) if $f_{X,c+\varepsilon}$ is non-decreasing for all $t > 0$ (f_X satisfies $\mathfrak{I}_{ND(0,\infty)}$).

Parts (a) and (b) together prove that L_1^+ satisfies $\mathfrak{I}_{ND(0,\infty)}$. Hence, by Lemma B1, $L_{(k)}^+$ satisfies $\mathfrak{I}_{ND(0,\infty)}$ with tilting parameter $c+\varepsilon$ for all $k \geq 2$. Since each term in the expansion (8.48) satisfies $\mathfrak{I}_{ND(0,\infty)}$ with the same tilting parameter $c+\varepsilon$, R^+ satisfies condition $\mathfrak{I}_{ND(0,\infty)}$ with tilting parameter $c+\varepsilon$.

Remaining proof of Theorem 4.

Since R^+ satisfies Proposition 1,

$$f_{R^+}(r) \sim \alpha_+ e^{-bt} \sim bS_{R^+}(t) \qquad t \to \infty,$$

where

$$\alpha_+ = -\operatorname{Res}\left\{\frac{\mathcal{M}_R(s)-e^{-B}}{1-e^{-B}};b\right\} = \frac{-\operatorname{Res}\left\{\mathcal{M}_R(s);b\right\}}{1-e^{-B}} = \frac{\alpha}{1-e^{-B}}$$

and

$$\alpha = \frac{e^{-B}}{(\Psi^+)'(b)}. \quad (8.54)$$

From this,

$$f_R(r) = (1-e^{-B})f_{R^+}(r) \sim \alpha e^{-bt} \qquad S_R(r) \sim f_R(r)/b \qquad t \to \infty.$$

B.4.3 *More general compound distributions*

Proof of Theorem 5. The proof uses much of the same arguments used to prove Theorem 3. Following that approach, we remove the point mass of R at 0 and consider the distribution of $R^+ = R|R > 0$ with MGF $[\mathcal{P}\{\mathcal{M}(s)\} - p(0)]/\{1 - p(0)\}$.

We now show that the convergence region for MGF $\mathcal{P}\{\mathcal{M}(s)\}$ is $\{\mathrm{Re}(s) < b\}$ where b is a dominant singularity of $\mathcal{P}\{\mathcal{M}(s)\}$. First, there is a unique value $b \in (0, c)$ solving $\mathcal{M}(s) = r$ since $\mathcal{M}(s)$ is strictly increasing for $s \in (0, c)$ and $1 = \mathcal{M}(0) < r < \mathcal{M}(c)$. To show b is a dominant singularity, we make use of the Darboux condition on $\mathcal{P}$ so that

$$\mathcal{P}\{\mathcal{M}(s)\} = g\{\mathcal{M}(s)\}\{r - \mathcal{M}(s)\}^{-w} + h\{\mathcal{M}(s)\}$$

$$= g\{\mathcal{M}(s)\}\mathcal{R}(s)^{-w}(b - s)^{-w} + h\{\mathcal{M}(s)\}, \qquad (8.55)$$

where

$$\mathcal{R}(s) = \frac{\mathcal{M}(b) - \mathcal{M}(s)}{b - s}$$

and principal branch values have been assumed for the multi-function $(r - s)^{-w}$ to allow for factorization. We now show that $g\{\mathcal{M}(s)\}$ and $h\{\mathcal{M}(s)\}$, are analytic on $\{\mathrm{Re}(s) \leq b\}$ and defer the proof that $\mathcal{R}(s)^{-w}$ is analytic to the next paragraph. For any $s = x + iy \in \{\mathrm{Re}(s) < b\}$,

$$|\mathcal{M}(s)| \leq \mathcal{M}(x) < \mathcal{M}(b) = r$$

so that $g\{\mathcal{M}(s)\}$ and $h\{\mathcal{M}(s)\}$ are analytic on $\{\mathrm{Re}(s) < b\}$. By compactness of the closed disc $\{|z| \leq r\}$, $g(z)$ and $h(z)$ are analytic on $\{|z| < r + \varepsilon\}$ for a sufficiently small $\varepsilon > 0$. For each boundary point $s_y = b + iy$ including $y = 0$, let N_y be a sufficiently small neighbourhood of s_y so that $|\mathcal{M}(s)| < r + \varepsilon/2$ for any $s \in N_y$. Then, $g\{\mathcal{M}(s)\}$ and $h\{\mathcal{M}(s)\}$ are analytic on the open cover $\cup_y N_y \cup \{\mathrm{Re}(s) < b\}$ for $\{\mathrm{Re}(s) \leq b\}$.

To show $\mathcal{R}(s)^{-w}$ is analytic on $\{\mathrm{Re}(s) \leq b\}$, it suffices to show that $\mathcal{R}$ maps $\{\mathrm{Re}(s) \leq b\}$ into a subset of C which does not overlap the branch cut from $-\infty$ to 0 used for the multifunction z^{-w}. Figure 1 shows images of the mapping $\mathcal{R}$ for vertical lines $\{x + iy : y \in (-\infty, \infty)\}$ for selected values of $x \leq b$ increasing from $x = -2, 0$, to $b = 1$. For increasing x, the images form expanding elliptically-looking closed curves that run counter clockwise from $\lim_{y \to -\infty} \mathcal{R}(x + iy) = 0 + 0i$ to $\lim_{y \to \infty} \mathcal{R}(x + iy) = 0 + 0i$. The image of $\{\mathrm{Re}(s) \leq b\}$ under $\mathcal{R}$ is therefore the closure of the dotted curve less 0 which is a set that avoids the branch cut of the multifunction. We now provide a more formal proof of this. Suppose the contrary, that the image of one of these curves crosses the branch cut at $-\varepsilon$ for $\varepsilon \geq 0$. Then, there exists $x \leq b$ and y such that

$$-\varepsilon = \mathcal{R}(x + iy) = \frac{\mathcal{M}(b) - \mathcal{M}(x + iy)}{b - x - iy}.$$

The value of y cannot be 0 since $\mathcal{R}(x) = \{\mathcal{M}(b) - \mathcal{M}(x)\}/(b - x) > 0$ for $x < b$ and $\mathcal{R}(b) = \mathcal{M}'(b) > 0$ since b is a removable singularity for $\mathcal{R}$. Thus, there must exist $x \le b$ and $y \ne 0$ such that

$$\mathcal{M}(x + iy) = \mathcal{M}(b) + \varepsilon(b - x) - i\varepsilon y.$$

This implies

$$|\mathcal{M}(x + iy)| = \sqrt{\{\mathcal{M}(b) + \varepsilon(b - x)\}^2 + (\varepsilon y)^2} > \mathcal{M}(b),$$

which is a contradiction. Therefore $\mathcal{R}(s)^{-w}$ is analytic on $\{\mathrm{Re}(s) \le b\}$. From the form in (8.55), $\mathcal{P}\{\mathcal{M}(s)\}$ satisfies $\mathfrak{I}_{\mathcal{M}}$ in Proposition 1 with

$$\lim_{s \to b}(b - s)^w \mathcal{P}\{\mathcal{M}(s)\} = g\{\mathcal{M}(b)\}\{\mathcal{M}'(b)\}^{-w}. \tag{8.56}$$

Since X_1 satisfies $\mathfrak{I}_{ND(0,\infty)}$ for some tilting parameter $c + \varepsilon$, each term in the tilted density

$$f_{R^+,c+\varepsilon}(t) = \frac{1}{1 - p(0)} \sum_{k=1}^{\infty} p(k) f_{X_1 + \cdots + X_k, c+\varepsilon}(t)$$

is also non-decreasing on $(0, \infty)$ so $f_{R^+}(t)$ satisfies $\mathfrak{I}_{ND(0,\infty)}$. Thus, using (8.56),

$$f_R(t) \sim \{1 - p(0)\}f_{R^+}(t) \sim \frac{g\{\mathcal{M}(b)\}}{\{\mathcal{M}'(b)\}^w \Gamma(w)} t^{w-1} e^{-bt} \qquad t \to \infty. \tag{8.57}$$

An asymptotic expansion for the stop-loss premium $L_R(t) = \int_t^{\infty} S_R(u)du$, follows from l'Hôpital's rule. Denote

$$\tilde{L}_R(t) = \int_t^{\infty} \tilde{S}_R(u)du = \int_t^{\infty} du \int_u^{\infty} \tilde{f}_R(v)dv,$$

where $\tilde{f}_R(v)$ is the gamma-like expansion for $f_R(v)$. Since $S_R(u)$ and $\tilde{S}_R(u)$ are continuous,

$$\lim_{t \to \infty} \frac{L_R(t)}{\tilde{L}_R(t)} = \lim_{t \to \infty} \frac{S_R(t)}{\tilde{S}_R(t)} = 1$$

as determined in (8.29). Since $\tilde{f}_R(t)$ and $\tilde{S}_R(t)$ are continuous, then, using l'Hôpital's rule again,

$$\lim_{t \to \infty} \frac{\tilde{S}_R(t)}{\tilde{L}_R(t)} = \lim_{t \to \infty} \frac{\tilde{f}_R(t)}{\tilde{S}_R(t)} = b$$

so that $L_R(t) \sim \tilde{L}_R(t) \sim \tilde{S}_R(t)/b \sim \tilde{f}_R(t)/b^2$ as $t \to \infty$.

Proof of Corollary 6. The proof follows the same approach as Theorem 5 so we provide only those details that differ. Using the Darboux conditions on $\mathcal{P}$ and $\mathcal{P}_X(e^b) = r$, then

$$\mathcal{P}\{\mathcal{M}(s)\} = g\{\mathcal{M}(s)\}\{\mathcal{P}_X(e^b) - \mathcal{P}_X(e^s)\}^{-w} + h\{\mathcal{M}(s)\} \tag{8.58}$$

$$= g\{\mathcal{M}(s)\}\mathcal{R}_1(s)^{-w}(e^b - e^s)^{-w} + h\{\mathcal{M}(s)\}, \tag{8.59}$$

where

$$\mathcal{R}_1(s) = \frac{\mathcal{P}_X(e^b) - \mathcal{P}_X(e^s)}{e^b - e^s}$$

and principal branch values are used for the multi-function $(e^b - e^s)^{-w}$. The same argument as in Theorem 5 shows that $\mathcal{R}_1$ maps $\{\mathrm{Re}(s) \le b\}$ into a subset of C that excludes the branch cut from $-\infty$ to 0. The remaining arguments as in Theorem 5 show that (8.59) satisfies Darboux condition $\mathfrak{D}_\mathcal{M}$ in Proposition 2. This justifies the expansions for $p_R(n)$ in (6.5) and for $S_R(n)$.

The expansion for the stop-loss premium follows from the Stolz–Cesàro theorem (Lemma A2, §A.5.1). Let $a_n = \sum_{k \ge n} S_R(k)$ and $b_n = \sum_{k \ge n} \tilde{S}_R(k) = (1 - e^{-b})^{-1} \sum_{k \ge n} \tilde{p}_R(k)$, where $\tilde{p}_R(k)$ is the asymptotic expansion for $p_R(k)$ in (6.5). Then

$$\lim_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n \to \infty} \frac{S_R(n)}{\tilde{S}_R(n)} = 1,$$

so $a_n \sim b_n$. Now $\sum_{k \ge n} \tilde{p}_R(k) \sim (1 - e^{-b})^{-1} \tilde{p}_R(n)$, as determined in the proof of Proposition 2, so that

$$a_n \sim b_n = (1 - e^{-b})^{-1} \sum_{k \ge n} \tilde{p}_R(k) \sim (1 - e^{-b})^{-2} \tilde{p}_R(n).$$

Thus, the stop-loss premium is $a_{n+1} \sim (1 - e^{-b})^{-2} \tilde{p}_R(n+1)$ and agrees with the result in Willmot (1989, thm. 2).

Proof of Theorem 6. We first show that $\mathcal{P}\{\mathcal{M}(s)\}$ is analytic on $\{\mathrm{Re}(s) \le b\} \subset C$ apart from an m-pole at b. For any $\mathbf{z} \in C^M$ such that $\mathrm{Re}(\mathbf{z}) \in \mathcal{O} \subset \Re^M$, note that

$$|\mathcal{P}(\mathbf{z})| = \left| E\left(\prod_{i=1}^M z_i^{N_i} \right) \right| \le E\left(\prod_{i=1}^M |z_i|^{N_i} \right) = \mathcal{P}(|\mathbf{z}|), \qquad (8.60)$$

where $|\mathbf{z}| = (|z_1|, ..., |z_M|)^T \in \Re^M$. Generating function $\mathcal{P}(\mathbf{z})$ is strictly increasing in each component of $\mathbf{z}$ for real $\mathbf{z} \in \mathcal{O} \cap (0, \infty)^M$. Thus, $\{\mathcal{M}(s) : 0 \le s < b\}$ traces a one-dimensional path through $\mathcal{O} \cap [1, \infty)^M$ with tangent directions $\{\mathcal{M}'(s) : 0 \le s < b\}$ such that $\mathcal{M}'(s) = \{\mathcal{M}'_1(s), \ldots \mathcal{M}'_M(s)\} > 0$ componentwise for each $s \in [0, b)$. The path crosses the boundary of set $\mathcal{O} \cap (0, \infty)^M$ at $\mathcal{M}(b) \in (1, \infty)^M$ so that $\mathcal{P}$ must be convergent on the open M-dimensional rectangle $(\mathbf{0}, \mathcal{M}(b)) \subset (0, \infty)^M$. Take $s = x + iy \in \{\mathrm{Re}(s) < b\}$. Since $|\mathcal{M}_j(s)| \le \mathcal{M}_j(x)$ for each j,

$$|\mathcal{P}\{\mathcal{M}(s)\}| \le \mathcal{P}\{|\mathcal{M}(s)|\} \le \mathcal{P}\{\mathcal{M}(x)\} < \infty,$$

since $\mathcal{M}(x) \in (\mathbf{0}, \mathcal{M}(b)) \subset (0, \infty)^M$. Thus, $\mathcal{P}\{\mathcal{M}(s)\}$ is analytic on $\{\mathrm{Re}(s) < b\}$. For the boundary, if $s_y = b + iy$ with $y \ne 0$, then $|\mathcal{M}_j(s_y)| < \mathcal{M}_j(b)$ for each j so that

$$|\mathcal{P}\{\mathcal{M}(s_y)\}| \le \mathcal{P}\{|\mathcal{M}(s_y)|\} < \infty$$

since $|\mathcal{M}(s_y)| \in (\mathbf{0}, \mathcal{M}(b)) \subset (0, \infty)^M$. We now construct an open neighbourhood of the boundary $\{s_y : y \neq 0\} \subset C$ on which $\mathcal{P}\{\mathcal{M}(s)\}$ is analytic. Let $N_y \subset C$ be an open neighbourhood of s_y sufficiently small so that $|\mathcal{M}(s)| < \{\mathcal{M}(b) + |\mathcal{M}(s_y)|\}/2$ componentwise for all $s \in N_y$. Then,

$$|\mathcal{P}\{\mathcal{M}(s)\}| \leq \mathcal{P}\{|\mathcal{M}(s)|\} < \mathcal{P}[\{\mathcal{M}(b) + |\mathcal{M}(s_y)|\}/2] < \mathcal{P}\{\mathcal{M}(b)\} = \infty$$

on $s \in N_y$, so $\mathcal{P}\{\mathcal{M}(s)\}$ is analytic on N_y. Thus, $\mathcal{P}\{\mathcal{M}(s)\}$ is analytic on $\cup_{y\neq 0} N_y \cup \{\mathrm{Re}(s) < b\}$ which is an open cover of $\{\mathrm{Re}(s) \leq b\}\backslash\{b\}$. Now, the function $g(s) = \mathcal{P}\{\mathcal{M}(s)\}(b - s)^m$ can be analytically extended to also include the point b since, based upon the assumptions of Theorem 6, it has a removable singularity at $s = b$. For this analytic extension, define

$$g(b) = \lim_{s \to b} \frac{\mathcal{N}\{\mathcal{M}(s)\}}{\mathcal{D}\{\mathcal{M}(s)\}/(b - s)^m} = \frac{\mathcal{N}\{\mathcal{M}(b)\}m!(-1)^m}{\partial^m \mathcal{D}\{\mathcal{M}(s)\}/\partial s^m|_{s=b}}, \qquad (8.61)$$

where the right-hand side results from using l'Hôpital's rule in the denominator m times. The use of l'Hôpital's rule is justified by the assumption that $\mathcal{D}$ is analytic at $\mathcal{M}(b)$. Thus, $g(s)$ is analytic on $\{\mathrm{Re}(s) \leq b\}$ and $\mathcal{P}\{\mathcal{M}(s)\} = g(s)(b - s)^{-m}$ so that R satisfies $\mathfrak{I}_{\mathcal{M}}$.

To show R^+ with MGF $[\mathcal{P}\{\mathcal{M}(s)\} - \mathcal{P}(\mathbf{0})]/\{1 - \mathcal{P}(\mathbf{0})\}$ satisfies $\mathfrak{I}_{ND(0,\infty)}$, take a Taylor expansion of $\mathcal{P}(\mathbf{z})$ about $\mathbf{z} = \mathbf{0}$ and evaluate it at $\mathbf{z} = \mathcal{M}(s)$. Each term in the Taylor expansion for $\mathcal{P}\{\mathcal{M}(s)\} - \mathcal{P}(\mathbf{0})$ contains at least one factor $\mathcal{M}_i(s)$ and the resulting summation is an infinite mixture of convolutions whose probability weights sum to $1 - \mathcal{P}(\mathbf{0})$. If $\mathcal{M}_i(s)$ satisfies $\mathfrak{I}_{ND(0,\infty)}$ with tilting parameter $c_i + \varepsilon_i$, then each addend in this mixture satisfies $\mathfrak{I}_{ND(0,\infty)}$ with tilting parameter $(c + \varepsilon)_* = \max_i\{c_i + \varepsilon_i\}$. Hence, R^+ satisfies $\mathfrak{I}_{ND(0,\infty)}$.

Now R^+ satisfies Ikehara conditions $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{ND(0,\infty)}$ and also Proposition 1, so that

$$f_R(t) \sim \beta t^{m-1} e^{-bt} \qquad t \to \infty$$

with

$$\beta = \frac{1}{(m-1)!} \lim_{s \to b} (b - s)^m \mathcal{P}\{\mathcal{M}(s)\} = \frac{g(b)}{(m-1)!}$$

and $g(b)$ given in (8.61). Thus β is given in (6.7).

Example 7. (Independent counts). A more general expansion, when b_1 is a singularity of non-integer order, results by expanding $f_R(t)$ as a finite mixture of 2^M terms, where terms are determined by which components in $\{R_i\}$ are positively valued and which ones are point masses at 0. The 2^{M-1} terms with component R_1^+ dominate with R_1^+ as an addend in each. Expansions for all these dominant terms follow if R_1^+ satisfies $\mathfrak{I}_{\mathcal{M}} \cap \mathfrak{I}_{ND(0,\infty)}$. This can be shown by using Theorem 5 if $\mathcal{P}_1$ and $\mathcal{M}_1$ satisfy its conditions so that MGF

$\mathcal{P}_1\{\mathcal{M}_1(s)\}$ has a singularity of order w_1 at $b_1 > 0$ in which $\mathcal{M}_1(b_1) = r_1 > 1$. Adding up all the dominant expansion contributions leads to

$$f_R(t) \sim \{1 - \mathcal{P}_1(0)\} \sum_{\mathcal{S}} \left[f_{R_1^+ + R_{\mathcal{S}}^+}(t) \prod_{i \in \mathcal{S}} \{1 - \mathcal{P}_i(0)\} \prod_{j \notin \mathcal{S}} \mathcal{P}_j(0) \right] \qquad (8.62)$$

as $t \to \infty$, where $R_{\mathcal{S}}^+ = \sum_{j \in \mathcal{S}} R_j^+$ and $\mathcal{S}$ ranges over all subsets of $\{2, \ldots, M\}$. Corollary 2 provides the expansion

$$f_{R_1^+ + R_{\mathcal{S}}^+}(t) \sim f_{R_1^+}(t) \prod_{i \in \mathcal{S}} \mathcal{M}_{R_i^+}(b_1)$$

$$= \{1 - \mathcal{P}_1(0)\}^{-1} f_{R_1}(t) \prod_{i \in \mathcal{S}} \left\{ \frac{\mathcal{M}_{R_i}(b_1) - \mathcal{P}_i(0)}{1 - \mathcal{P}_i(0)} \right\}.$$

Substituting into (8.62) leads to

$$f_R(t) \sim f_{R_1}(t) \sum_{\mathcal{S}} \left[\prod_{i \in \mathcal{S}} \{\mathcal{M}_{R_i}(b_1) - \mathcal{P}_i(0)\} \prod_{j \notin \mathcal{S}} \mathcal{P}_j(0) \right]$$

$$= f_{R_1}(t) \prod_{i=2}^{M} [\{\mathcal{M}_{R_i}(b_1) - \mathcal{P}_i(0)\} + \mathcal{P}_i(0)]$$

$$= f_{R_1}(t) \prod_{i=2}^{M} \mathcal{M}_{R_i}(b_1). \qquad (8.63)$$

Theorem 5 provides an expansion for $f_{R_1}(t)$ which, when substituted into (8.63), leads to the expansion for $f_R(t)$ in (6.8).

Proof of Theorem 7. The proof is almost identical to that of Theorem 6 so only differences are noted. The main difference is that the weaker assumptions for Proposition 2 must hold rather than those for Proposition 1.

When showing that $\mathcal{P}\{\mathcal{M}(s)\}$ is analytic on $\{\mathrm{Re}(s) \leq b\}$ apart from an m-pole at b, the argument that it is analytic on the boundary $\{\mathrm{Re}(s) = b\}$ apart from $s = b$ needs modification. The part of the argument given in Theorem 6 that needs modification with integer-valued $\{X_j\}$ is that given for justifying that $|\mathcal{M}_j(b + iy)| < \mathcal{M}_j(b)$ with strict inequality for each j when $y \in (-\pi, \pi] \backslash \{0\}$. This justification holds for absolutely continuous distributions, as shown by Daniels (1954, p. 632), but it remains to be shown for an integer-valued mass function $p_j(n)$. Let $b + iy$ be on the boundary of the principal convergence region of $\mathcal{P}\{\mathcal{M}(s)\}$ with $y \in (-\pi, \pi] \backslash \{0\}$. The proof is by contradiction so assume equality which means that $\mathcal{M}_j(b + iy) = \mathcal{M}_j(b)e^{i\alpha}$ for some $\alpha \in (-\pi, \pi]$. Since

$\mathcal{M}_j$ is analytic at $b + iy$ and b, then

$$0 + 0i = \mathcal{M}_j(b + iy) - \mathcal{M}_j(b)e^{i\alpha} = \sum_{n \geq 0} e^{bn} p_j(n) \left[\{\cos(yn) - \cos\alpha\} + i\{\sin(yn) - \sin\alpha\} \right]$$

$$= A + iB.$$

Thus,

$$0 = A\cos\alpha + B\sin\alpha = \sum_{n \geq 0} e^{bn} p_j(n)\{\cos(yn - \alpha) - 1\} \tag{8.64}$$

which is a contradiction since $\cos(yn - \alpha) - 1$ must be negative for some $n \geq 0$ for which $p_j(n) > 0$. To see this, note that for (8.64) to hold, we require that $\cos(yn - \alpha) = 1$ for a.e. $n \geq 0$. This can only happen when a.e. value $yn - \alpha \in \{0, \pm 2\pi, \ldots\}$. Thus the values of $yn - \alpha$ must be either identically $-\alpha$ (so $y = 0$), or they must be spaced $2\pi k$ apart for integer k which makes $y = \pm 2\pi k$. In either case, this cannot be and we reach a contradiction.

We now show that $\mathcal{P}\{\mathcal{M}(s)\}$ satisfies Darboux condition $\mathfrak{D}_{\mathcal{M}}$ in (4.3) of Proposition 2. Since b is an m-zero of $\mathcal{D}\{\mathcal{M}(s)\}$, we can use l'Hôpital's rule m times to show that

$$\lim_{s \to b} \frac{\mathcal{D}\{\mathcal{M}(s)\}}{(e^b - e^s)^m} = \frac{\partial^m \mathcal{D}\{\mathcal{M}(s)\}/\partial s^m|_{s=b}}{m!(-1)^m e^{bm}}.$$

Thus, we may analytically continue $g(e^s) = \mathcal{P}\{\mathcal{M}(s)\}(e^b - e^s)^m$ to b by defining

$$g\left(e^b\right) = \lim_{s \to b} \frac{\mathcal{N}\{\mathcal{M}(s)\}}{\mathcal{D}\{\mathcal{M}(s)\}/(e^b - e^s)^m} = \frac{\mathcal{N}\{\mathcal{M}(b)\}m!(-1)^m e^{bm}}{\partial^m \mathcal{D}\{\mathcal{M}(s)\}/\partial s^m|_{s=b}}.$$

Thus, $g(e^s)$ is analytic on $\{\mathrm{Re}(s) \leq b\}$ and the compound sum R satisfies the conditions of Proposition 2.

From (4.2), the value β is determined as

$$\beta = \frac{g\left(e^b\right)e^{-bm}}{(m-1)!} = \frac{\mathcal{N}\{\mathcal{M}(b)\}m(-1)^m}{\partial^m \mathcal{D}\{\mathcal{M}(s)\}/\partial s^m|_{s=b}}.$$

B.5 Proofs for first-passage distributions in semi-Markov processes

B.5.1 *Proof of Proposition 3*

The jump chain for the SMP of relevant states concerned with passage $1 \to M$ has a transition probability matrix $\mathbf{P}$ and we can modify it to make state M absorbing. Let $\tilde{\mathbf{P}}$ denote $\mathbf{P}$ with all Mth row entries set to 0. Let $\xi_1 = (1, 0, \ldots, 0)^T$ and $\xi_M = (0, \ldots 0, 1)^T$ be $M \times 1$ indicator vectors. Denote by $Y = \sum_{i,j=1}^M N_{ij} = n \geq 1$ the total number of steps required for first-passage to state M so that $\xi_1^T \tilde{\mathbf{P}}^n \xi_M = P(Y = n)$ and

$$\xi_1^T (\tilde{\mathbf{P}} \odot \mathbf{Z})^n \xi_M = E\left\{ \prod_{i=1}^M \prod_{j=1}^M z_{ij}^{N_{ij}} 1_{\{Y=n\}} \right\}. \tag{8.65}$$

In this expression, since state M is absorbing, $N_{Mj} = 0$ w.p. 1 for $j \geq 1$ so that (8.65) does not depend on $\{z_{Mj} : j \in \mathcal{S}\}$. Summing over $n \geq 1$, then

$$\xi_1^T (\mathbf{I}_M - \tilde{\mathbf{P}} \odot \mathbf{Z})^{-1} \xi_M = \xi_1^T \{ (\mathbf{I}_M - \tilde{\mathbf{P}} \odot \mathbf{Z})^{-1} - \mathbf{I}_M \} \xi_M = E \left\{ \prod_{i=1}^M \prod_{j=1}^M z_{ij}^{N_{ij}} 1_{\{Y < \infty\}} \right\} \tag{8.66}$$

on $\{\mathbf{Z} \in \Re^{M^2} : |\lambda_1(\tilde{\mathbf{P}} \odot \mathbf{Z})| < 1\}$, where $\lambda_1(\cdot)$ denotes the eigenvalue with the largest modulus for the matrix argument. The leftmost side of (8.66) is the $(1, M)$ component of the inverse of $\mathbf{I}_M - \tilde{\mathbf{P}} \odot \mathbf{Z}$ so

$$\frac{(M, 1)\text{-cofactor of } \{\mathbf{I}_M - \tilde{\mathbf{P}} \odot \mathbf{Z}\}}{|\mathbf{I}_M - \tilde{\mathbf{P}} \odot \mathbf{Z}|} = E \left\{ \prod_{i=1}^M \prod_{j=1}^M z_{ij}^{N_{ij}} 1_{\{Y < \infty\}} \right\}. \tag{8.67}$$

Since the last row of matrix $\mathbf{I}_M - \tilde{\mathbf{P}} \odot \mathbf{Z}$ is ξ_M^T, then $|\mathbf{I}_M - \tilde{\mathbf{P}} \odot \mathbf{Z}| = (M, M)$-cofactor of $\mathbf{I}_M - \tilde{\mathbf{P}} \odot \mathbf{Z}$. Furthermore, the $(M, 1)$- and (M, M)-cofactors of $\mathbf{I}_M - \tilde{\mathbf{P}} \odot \mathbf{Z}$ do not depend on the Mth row of $\tilde{\mathbf{P}} \odot \mathbf{Z}$; thus they agree with the $(M, 1)$- and (M, M)-cofactors of $\mathbf{I}_M - \mathbf{P} \odot \mathbf{Z}$. Therefore, substituting into the left side of (8.67), it becomes

$$\frac{(M, 1)\text{-cofactor of } \{\mathbf{I}_M - \mathbf{P} \odot \mathbf{Z}\}}{(M, M)\text{-cofactor of } \{\mathbf{I}_M - \mathbf{P} \odot \mathbf{Z}\}} = E \left\{ \prod_{i=1}^M \prod_{j=1}^M z_{ij}^{N_{ij}} 1_{\{Y < \infty\}} \right\}. \tag{8.68}$$

Up to this point, the equality in both (8.67) and (8.68) holds on

$$\mathcal{O} = \{\mathbf{Z} \in \Re^{M^2} : |\lambda_1(\tilde{\mathbf{P}} \odot \mathbf{Z})| < 1\} = \{\mathbf{Z} : |\lambda_1\{(\mathbf{P} \odot \mathbf{Z})_{\backslash MM}\}| < 1\}.$$

The right-hand side of (8.68) is convergent on $\mathcal{R} = \{\mathbf{Z} : |z_{ij}| \leq 1 \text{ for } i, j = 1, \ldots M\}$ and values are dominated above by the value $P(Y < \infty)$ achieved at $\mathbf{Z} = \mathbf{1}$, a matrix of ones. Thus, we conclude that $\mathcal{O} \supset \mathcal{R}$ and this dominating value is

$$\frac{(-1)^{M+1} |\mathbf{\Psi}_{M1}(0)|}{|\mathbf{\Psi}_{MM}(0)|} = \frac{(M, 1)\text{-cofactor of } \{\mathbf{I}_M - \mathbf{P}\}}{(M, M)\text{-cofactor of } \{\mathbf{I}_M - \mathbf{P}\}} = P(Y < \infty). \tag{8.69}$$

Since the left-hand side of (8.69) is $f_{1M} = P(X < \infty)$, our derivation has proved that $P(X < \infty) = P(Y < \infty)$, however, it is also worth clarifying this. First, if the jump chain of the SMP arrives at state M in a finite number of steps, then the sojourn time must also be finite, so $P(Y < \infty) \leq P(X < \infty)$. Also, since the state space $\mathcal{S}$ is finite, the SMP is non-explosive (Heyman and Sobel, 1982, p. 323); hence any sojourn from $1 \to M$ which is achieved in finite time X must use only a finite number of state transitions so that $P(X < \infty) \leq P(Y < \infty)$. Thus, $P(Y < \infty) = P(X < \infty) = f_{1M}$. Dividing (8.68) by (8.69) confirms that (7.2) is the conditional PGF of $\mathbf{N}$ given $Y < \infty$.

B.5.2 *Proof of Theorem 8*

We confirm that assumptions (i)–(iv) in §7 ensure that the conditions of Theorem 6 hold for the compound distribution of passage time $X|X < \infty$ with MGF $\mathcal{F}_{1M}(s) = \mathcal{P}\{\mathcal{M}(s)|Y < \infty\}$.

Under assumption (i), $\mathcal{P}(\mathbf{Z}|Y < \infty)$ is convergent on $\{\mathbf{Z} : |z_{ij}| \leq 1 \text{ for } i, j = 1, \ldots M\}$ as required by Theorem 6. Conditions (ii) and (iii) ensure all other conditions of Theorem 6 apart from the assumption that all densities $\{g_{ij}(t)\}$ from the first $M-1$ rows of $\mathbf{K}(t)$ satisfy $\mathfrak{I}_{ND(0,\infty)}$. This can be weakened to assumption (iv) which assumes that all members of a blockade $\mathfrak{B}$ satisfy $\mathfrak{I}_{ND(0,\infty)}$. To see why, expand $\mathcal{P}(\mathbf{Z}|Y < \infty)$ in a Taylor series about $\mathbf{Z} = \mathbf{0}$ and then substitute $\mathbf{Z} = \mathcal{M}(s)$. The leading term is $P(\mathbf{N} = \mathbf{0}|Y < \infty) = 0$ and all other addends represent mutually exclusive transmittance pathways for the sojourn from $1 \to M$. In particular, if $\mathfrak{P}$ is the set of all distinct ordered pathways from $1 \to M$, this expansion yields

$$\mathcal{F}_{1M}(s) = \mathcal{P}\{\mathcal{M}(s)|Y < \infty\} = \frac{1}{f_{1M}} \sum_{\mathfrak{p} \in \mathfrak{P}} p_{\mathfrak{p}} \mathcal{M}_{\mathfrak{p}}(s), \qquad (8.70)$$

where $p_{\mathfrak{p}}$ denotes the probability the sojourn takes path $\mathfrak{p}$ and $\mathcal{M}_{\mathfrak{p}}(s)$ is the product of MGFs for the sequence of ordered state transitions that characterise path $\mathfrak{p}$. Each term $\mathcal{M}_{\mathfrak{p}}(s)$ has at least one factor which is a member of the blockade $\mathcal{B}$; thus, the density corresponding to $\mathcal{M}_{\mathfrak{p}}(s)$ satisfies $\mathfrak{I}_{ND(0,\infty)}$. All terms on the right of (8.70) have densities that satisfy $\mathfrak{I}_{ND(0,\infty)}$ with a common tilting parameter so the density of $\mathcal{F}_{1M}(s)$ also satisfies $\mathfrak{I}_{ND(0,\infty)}$ by the same arguments used previously for such infinite expansions.

To find β, note that

$$f_{1M} = f_{1M}\mathcal{F}_{1M}(0) = \frac{(-1)^{M+1}|\mathbf{\Psi}_{M1}(0)|}{|\mathbf{\Psi}_{MM}(0)|}.$$

Therefore, from (6.7),

$$\beta = (-1)\frac{(-1)^{M+1}|\mathbf{\Psi}_{M1}(b)|}{f_{1M}\,\partial|\mathbf{\Psi}_{MM}(s)|/\partial s|_{s=b}}.$$

Now, use matrix calculus to compute $\partial|\mathbf{\Psi}_{MM}(s)|/\partial s = \text{tr}[\text{adj}\{\mathbf{\Psi}_{MM}(s)\}\dot{\mathbf{\Psi}}_{MM}(s)]$ which gives β in (7.4).

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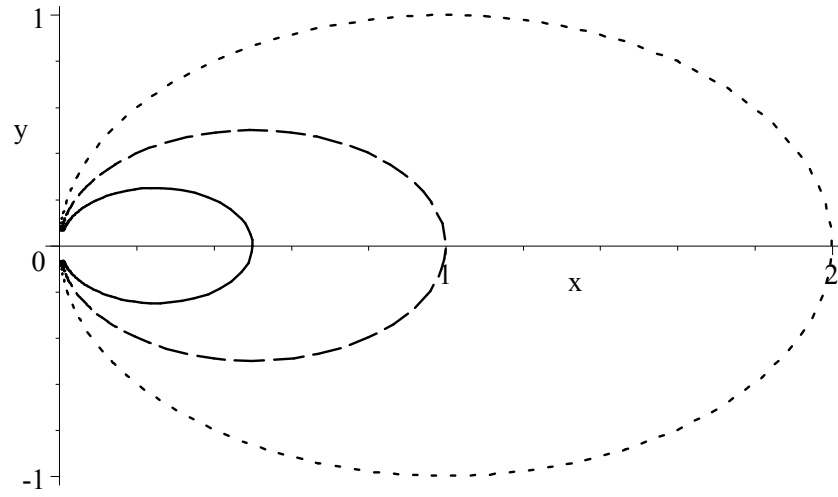
Figure 1. Images of the vertical lines $\{x + iy : y \in (-30, 30)\}$ for $x = -2$ (solid), $x = 0$ (dashed) and $x = 1$ (dotted) under the mapping $\mathcal{R} : C \to C$ when $\mathcal{M}(s) = (1 - s/2)^{-1}$ and $b = 1$.