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I. INTRODUCTION

A Markov Renewal Process (MRP) with $m(< \infty)$ states is one which records at each time t, the number of times a system visits each of the m states up to time t, if the system moves from state to state according to a Markov chain with transition probability matrix $P_0 = [p_{ij}]$ and if the time required for each successive move is a random variable whose distribution function (d.f.) depends on the two states between which the move is made. Thus, if the system moves from state i to state j, the holding time in the state i has $F_{ij}(x)$ as its d.f. $(i,j = 1, 2, \cdots, m)$. We set $Q_{ij}(x) = p_{ij}F_{ij}(x)$. Obviously $Q_{ij}(x) = 0$ for $x < 0$ and $\sum_{j=1}^{m} Q_{ij}(\infty) = 1$. Let $N_j(t)$ denote the number of times the system visits state j in the time interval $(0,t)$ and let $f_{ij}(t)$ denote the number of transitions from i to j, in the interval $(0,t)$. $N(t) = \sum_{j=1}^{m} N_j(t)$ is the total number of transitions of the MRP. Let Z_t denote the state of the system at time t and let $J_0, J_1, J_2, \cdots,$ be the successive states assumed by the MRP. Further we shall denote by $X_0(= 0), X_1, X_2, \cdots,$ the times spent by the MRP in the successive states. Thus, for example, the MRP makes a transition from J_{k-1} to J_k, after remaining in J_{k-1} for time X_k.

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Pyke (1961a, 1961b, 1964, 1968) has studied Markov Renewal Processes extensively and his notation is followed in this paper as far as possible. The matrix $F(t) = [f_{ij}(t)]$ is called the transition count matrix of the MRP. The distribution of the transition count matrix of a Markov chain was first obtained by Whittle (1955) and later by Dawson and Good (1957), Goodman (1958), and Billingsley (1961). In this paper, the distribution of the transition count matrix of a Markov renewal process is obtained and the first and second order moments of this distribution are also derived. Asymptotic expressions for these moments, for large t are also obtained.

If g is a real measurable function of the two states between which a transition is made by the MRP and also of the time taken to make this transition and if we set

$$W_g(t) = \sum_{n=1}^{N(t)} g(J_{n-1}, J_n, X_n), \tag{1.1}$$

the process $\{W_g(t), t > 0\}$ is called a cumulative process corresponding to the MRP. Pyke and Schaufele (1964), (see also Moore and Pyke, 1968), have obtained the mean and variance of $W_g(t)$. In this paper, we derive the entire distribution of $W_g(t)$. The moments of $W_g(t)$ and also the covariance between $W_g(t)$ and another process $W_h(t)$ are obtained in this paper by an alternative method which yields expressions which are easier to calculate than Pyke's as these moments are expressed directly in terms of the basic quantities Q_{ij} and p_{ij} of the MRP and not in terms of recurrence times of the MRP and the imbedded Markov chain, as Pyke has done.

II. TRANSITION COUNT MATRIX

Every transition into a state k must be followed by an exit out of state k , except for the initial state, say i , and the final state, say j . It follows, therefore, that

$$f_{k\cdot}(t) - f_{\cdot k}(t) = \delta_{ik} - \delta_{jk} \quad , \quad (k = 1, 2, \cdots , m) \tag{2.1}$$

where $f_{k\cdot}(t) = \sum_{\ell=1}^{m} f_{k\ell}(t)$, $f_{\cdot k}(t) = \sum_{\ell=1}^{m} f_{\ell k}(t)$ and δ_{ik} , δ_{jk} are all Kronecker deltas. Let $F = [f_{ij}]$, where f_{ij} $(i,j = 1, 2, \cdots , m)$ are all non-negative integers. Let

$$V_{ij}(F,t) = \mathrm{Prob}\left(F(t) = F \quad , \quad Z_t = j \,|\, Z_0 = i\right) \tag{2.2}$$

In other words, $V_{ij}(F,t)$ is the joint distribution of the transition count matrix and the final state of the MRP, conditional on the initial state being i . Obviously, if F does not satisfy (2.1), $V_{ij}(F,t) = 0$. Similarly $F = 0$ implies that there are no transitions of the MRP in $(0,t)$ and so, the initial and final states must be the same. Hence

$$V_{ij}(0,t) = \begin{cases} 0 \text{ if } j \neq i \\ 1 - \sum_{k=1}^{m} Q_{ik}(t) \end{cases} \tag{2.3}$$

Observe also that $F = 0$ satisfies (2.1) only when $i = j$. Again,

$$V_{ij}(F,t) = 0 \quad , \text{ if } F \text{ does not satisfy (2.1)} . \tag{2.4}$$

But, if $F \neq 0$ and F satisfies (2.1), we can consider the first transition and the subsequent transition count, yielding

$$V_{ij}(F,t) = \sum_{k=1}^{m} Q_{ik}(t) * V_{kj}\left(F(i,k),\ t\right) \tag{2.5}$$

where * denotes convolution and $F(i,k)$ denotes the matrix obtained from

F by reducing its entry in the i^{th} row and k^{th} column by unity. The generating function of the probabilities $V_{ij}(F,t)$ is therefore,

$$\Phi_{ij}(\xi,t) = \sum_{F} \prod_{k,\ell=1}^{m} \xi_{k\ell}^{f_{k\ell}} V_{ij}(F,t) \tag{2.6}$$

where, $\xi = [\xi_{ij}]$. From (2.3), (2.4), and (2.5),

$$\Phi_{ij}(\xi,t) = \sum_{k=1}^{m} Q_{ik}(t) * \xi_{ik}\Phi_{kj}(\xi,t) + \delta_{ij}\{1 - \sum_{k=1}^{m} Q_{ik}(t)\} . \tag{2.7}$$

Using

$$q_{ij}(s) = \int_0^\infty e^{-st} d_t Q_{ij}(t) \quad , \tag{2.8}$$

and

$$\Psi_{ij}^0(\xi,s) = \int_0^\infty e^{-st} d_t \Phi_{ij}(\xi,t) \quad , \tag{2.9}$$

to denote the Laplace-Steiltjes transforms (L. - S.T.) of Q_{ij} and Φ_{ij}, respectively, and denoting by $q(s)$ and $\Psi^0(\xi,s)$, the matrices $[q_{ij}(s)]$ and $[\Psi_{ij}^0(\xi,s)]$, respectively, we obtain from (2.7),

$$\Psi^0(\xi,s) = q(s) \square \xi \cdot \Psi^0(\xi,s) + I - h(s) \quad , \tag{2.10}$$

where

$$q(s) \square \xi = [q_{ij}(s)\xi_{ij}] \quad ,$$

$$h_i(s) = \sum_{k=1}^{m} q_{ik}(s) \quad ,$$

and

$$h(s) = [\delta_{ij}h_i(s)] \quad ,$$

Hence,

$$\Psi^0(\xi,s) = [I - q(s) \square \xi]^{-1}[I - h(s)] \tag{2.11}$$

Whittle (1955) has proved the following result:

Whittle's lemma: Coefficient of $\prod_{k,\ell=1}^{m} a_{k\ell}^{f_{k\ell}}$ in the $(i,j)^{th}$ element of the $m \times m$ matrix $(I - A)^{-1}$, where $A = [a_{ij}]$, is

$$F^*_{ji} \frac{\prod_{k=1}^{m} f_{k\cdot}!}{\prod_{k,\ell=1}^{m} f_{k\ell}!} \tag{2.12}$$

if $f_{k\ell}$ are non-negative integers such that

$$f_{k\cdot} - f_{\cdot k} = \delta_{ik} - \delta_{jk} \quad , \quad (k = 1, 2, \cdots , m) ,$$

and is zero otherwise. Here F^*_{ji} is the cofactor of the $(j,i)^{th}$ element of the matrix $F^* = [f^*_{ij}]$, defined by

$$f^*_{ij} = \begin{cases} \delta_{ij} - f_{ij}/f_{i\cdot} & ; \quad f_{i\cdot} > 0 \\ \delta_{ij} & , \quad f_{i\cdot} = 0 \end{cases} \tag{2.13}$$

Hence, the L. - S.T. of $V_{ij}(F,t)$ viz. the coefficient of $\prod_{k,\ell=1}^{m} \xi_{k\ell}^{f_{k\ell}}$ in $\Psi_{ij}(\xi,s)$ is

$$\begin{cases} F^*_{ji} \dfrac{\prod_{k=1}^{m} f_{k\cdot}!}{\prod_{k,\ell=1}^{m} f_{k\ell}!} \prod_{k,\ell=1}^{m} \left(q_{k\ell}(s)\right)^{f_{k\ell}} \left(1 - h(s)\right) \\ \qquad \text{if } F \text{ satisfies (2.1)} \\ 0 \quad \text{otherwise.} \end{cases} \tag{2.14}$$

The distribution of $F(t)$ alone can be obtained from (2.11) and (2.14) by summing with respect to the fixed state j . Thus, if

$$U_i(F,t) = \text{Prob}\left(F(t) = F \mid Z_0 = i\right) \quad , \quad i = 1, 2, \cdots , m \tag{2.15}$$

the L. - S.T. of the generating function of $U_i(F,t)$ is given by $\Psi_i(\xi,s)$, the i^{th} element of the column vector

$$\begin{aligned} \underline{\Psi}(\xi,s) &= \Psi^0(\xi,s)\underline{e} \\ &= \left(I - q(s) \,\square\, \xi\right)^{-1}\left(I - h(s)\right)\underline{e} \\ &= \left(I - q(s) \,\square\, \xi\right)^{-1}\left(I - q(s)\right)\underline{e} \quad . \end{aligned} \tag{2.16}$$

where $\underline{e}' = [1, 1, \cdots , 1]$. The L. - S.T. of $U_i(F,t)$ is, from (2.14),

$$F^*_{ki} \frac{\prod_{\alpha=1}^{m} f_{\alpha\cdot}!}{\prod_{\alpha,\beta=1}^{m} f_{\alpha\beta}!} \prod_{\alpha,\beta=1}^{m} \left(q_{\alpha\beta}(s)\right)^{f_{\alpha\beta}} \left(1 - h_k(s)\right) \quad , \tag{2.17}$$

if the matrix F is such that, the equations

$$f_{\alpha\cdot} - f_{\cdot\alpha} = \delta_{i\alpha} - \delta_{k\alpha} \tag{2.18}$$

hold for all $\alpha = 1, 2, \cdots , m$. If there is no such k , for which (2.18) hold, $U_i(F,t)$ is zero.

III. MOMENTS OF F(t)

Let $F(t \mid Z_0 = i)$ denote the transition count matrix of the MRP in $(0,t)$, given that the initial state is i . Let further,

$$\begin{aligned} \sigma_{\alpha\beta}(i,s) &= \text{L. - S.T. of } E\{f_{\alpha\beta}(t) \mid Z_0 = i\} \quad , \\ \sigma_{\alpha\beta\gamma\delta}(i,s) &= \text{L. - S.T. of } E\{f_{\alpha\beta}(t) f_{\gamma\delta}(t) \mid Z_0 = i\} \quad , \end{aligned} \tag{3.1}$$

and let $\underline{\sigma}_{\alpha\beta}(s)$, $\underline{\sigma}_{\alpha\beta\gamma\delta}(s)$ be the column vectors of $\sigma_{\alpha\beta}(i,s)$ and

$\sigma_{\alpha\beta\gamma\delta}(i,s)$, $(i = 1, 2, \cdots, m)$, respectively. We can obtain the L. - S.T. of the moments of $f_{\alpha\beta}(t)$ by differentiating (2.11), an appropriate number of times with respect to the appropriate $\xi_{\alpha\beta}$'s and then putting $\xi = I$. However, we use a different method here. This method was employed by Martin (1967) for the moments of the transition count matrix of a Markov chain. We extend it here to an M.R.P. Observe that

$$f_{\alpha\beta}(t|Z_0 = i) = \begin{cases} \delta_{i\alpha}\delta_{k\beta} + f_{\alpha\beta}(t - X_1|Z_0 = k), \text{ if } J_1 = k, \ (k = 1, 2, \cdots m) \\ 0 \quad , \text{ if there is no transition in } (0,t) \end{cases} \tag{3.2}$$

Taking L. - S.T. of the expectation on both sides,

$$\begin{aligned} \sigma_{\alpha\beta}(i,s) &= \sum_{k=1}^{m} q_{ik}(s)\delta_{i\alpha}\delta_{k\beta} + \sum_{k=1}^{m} q_{ik}(s)\sigma_{\alpha\beta}(k,s) \\ &= q_{i\beta}(s)\delta_{i\alpha} + \sum_{k=1}^{m} q_{ik}(s)\sigma_{\alpha\beta}(k,s) \end{aligned} \tag{3.3}$$

Hence,

$$\underline{\sigma}_{\alpha\beta}(s) = q_{\alpha\beta}(s)\underline{e}_{\alpha} + q(s)\underline{\sigma}_{\alpha\beta}(s) \tag{3.4}$$

where $\underline{e}_r$ denotes an $m \times 1$ vector, with zeroes everywhere, except the r^{th} $(r = 1, 2, \cdots, m)$ element which is unity. Finally, therefore,

$$\begin{aligned} \underline{\sigma}_{\alpha\beta}(s) &= q_{\alpha\beta}(s)\left(I - q(s)\right)^{-1} \underline{e}_{\alpha} \\ &= q_{\alpha\beta}(s) \cdot \alpha^{\text{th}} \text{ column of } \left(I - q(s)\right)^{-1} \end{aligned} \tag{3.5}$$

or

$$\sigma_{\alpha\beta}(i,s) = q_{\alpha\beta}(s)\left\{\left(I - q(s)\right)^{-1}\right\}_{i\alpha} . \tag{3.6}$$

where $\{B\}_{ij}$ denotes the element in the i^{th} row and j^{th} column of a matrix B . In exactly the same way,

$$f_{\alpha\beta}(t|Z_0 = i)f_{\gamma\delta}(t|Z_0 = i) = \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{\alpha i}\delta_{\beta k} + \delta_{i\gamma}\delta_{k\delta}f_{\alpha\beta}(t - X_1|Z_0 = k)$$

$$+ \delta_{i\alpha}\delta_{k\beta}f_{\gamma\delta}(t - X_1|Z_0 = k) \tag{3.7}$$

$$+ f_{\alpha\beta}(t - X_1|Z_0 = k)f_{\gamma\delta}(t - X_1|Z_0 = k) \quad ,$$

$$\text{if } J_1 = k \quad , \quad (k = 1, 2, \cdots , m)$$

and is zero, if there is no transition in $(0,t)$. Taking L. - S.T. of expectations on both sides of (3.7), and combining all such results for $i = 1, 2, \cdots , m$, we obtain after a little algebra,

$$\underline{\sigma}_{\alpha\beta\gamma\delta}(s) = \delta_{\alpha\gamma}\delta_{\beta\delta}\underline{\sigma}_{\alpha\beta}(s) + \sigma_{\alpha\beta}(\delta,s)\underline{\sigma}_{\gamma\delta}(s) + \sigma_{\gamma\delta}(\beta,s)\underline{\sigma}_{\alpha,\beta}(s) \tag{3.8}$$

IV. ASYMPTOTIC VALUES OF THE MOMENTS OF F(t)

The moments of $f_{\alpha\beta}(t)$, derived in Section III, are hidden under the Laplacian curtain and in this section, we derive asymptotic expressions for them, for large t , by expanding the L. - S.T.'s in powers of s and using Tauberian arguments, as employed by Cox (1962), in the case of ordinary renewal processes.

Kshirsagar and Gupta (1967) have proved that

$$\left(I - q(s)\right)^{-1} = \frac{1}{sk_1} H_0 + a\, H_0 + \frac{1}{k_1} H_1 + O(1) \tag{4.1}$$

where

$$H_0 = \text{adjoint of } I - P_0 = \underline{e}\,\underline{d}' \quad ,$$

$$\underline{d} \text{ being a column vector such that } \underline{d}'P_0 = \underline{d}' \quad , \tag{4.2}$$

$$H_1 = [h_{ij}] = \text{coefficient of } s \text{ in the adjoint of } I - P_0 + sP_1 \quad , \tag{4.3}$$

$$q(s) = P_0 - sP_1 + 0(s) \tag{4.4}$$

$$P_1 = [c_{ij}] \quad , \tag{4.5}$$

$$c_{ij} = p_{ij} \int_0^\infty x dF_{ij}(x) \quad , \tag{4.6}$$

$$a = \frac{1}{k_1}\left(\frac{k_2}{2k_1} - a_1\right) \quad , \tag{4.7}$$

$$k_1 = \underline{d}' P_1 \underline{d} \tag{4.8}$$

$$k_2 = \underline{d}' P_2 \underline{d} \tag{4.9}$$

$$a_1 = \frac{\text{sum of the principal minors of order m-2 of } P_1^{-1}(I-P_0)}{\text{sum of the principal minors of order m-1 of } P_1^{-1}(I-P_0)} \tag{4.10}$$

Observe that the vector $\underline{d}$, defined by (4.2) is proportional to the vector of stationary probabilities of the imbedded Markov chain, the transition probability matrix of which is P_0 . It should also be noted that Kshirsagar and Gupta use α instead of $\frac{1}{k_1}$ and define it as the reciprocal of the product of $|P_1|$ and the non-zero latent roots of $P_1^{-1}(I-P_0)$; but it can be proved that this is the same as (4.8).

Using (4.1) and (4.4), it can be readily seen that

$$\sigma_{\alpha\beta}(i,s) = \frac{1}{sk_1} d_\alpha p_{\alpha\beta} - \frac{d_\alpha c_{\alpha\beta}}{k_1} + ad_\alpha p_{\alpha\beta} + \frac{1}{k_1} h_{i\alpha} p_{\alpha\beta} + 0(1) \tag{4.11}$$

and hence,

$$\frac{1}{t} E(f_{\alpha\beta} | Z_0 = i) \sim \frac{d_\alpha p_{\alpha\beta}}{k_1} \quad . \tag{4.12}$$

where $\sim$ denotes asymptotic equivalence. Using (4.11), after a little algebra, one obtains

$$\frac{1}{t}\,\mathrm{Cov}(f_{\alpha\beta}\,,\,f_{\gamma\delta}|Z_0 = i) \sim \delta_{\alpha\gamma}\delta_{\beta\delta}d_\alpha p_{\alpha\beta}/k_1 - \frac{d_\alpha d_\gamma p_{\alpha\beta}c_{\gamma\delta}}{k_1^2}$$

$$- \frac{d_\alpha d_\gamma p_{\gamma\delta}c_{\alpha\beta}}{k_1^2} + \frac{2ad_\alpha d_\gamma p_{\alpha\beta}p_{\gamma\delta}}{k_1}$$

$$+ \frac{d_\gamma h_{\delta\alpha}p_{\alpha\beta}p_{\gamma\delta}}{k_1^2} + d_\alpha h_{\beta\gamma}\,\frac{p_{\alpha\beta}p_{\gamma\delta}}{k_1^2} \qquad (4.13)$$

V. DISTRIBUTION OF $W_g(t)$

$W_g(t)$ has already been defined in (1.1). In this section, we obtain the Laplace transform of its distribution. Let

$$\xi_{ij}(\eta) = \int_0^\infty e^{-\eta g(i,j,x)}\, dQ_{ij}(x) \qquad (5.1)$$

It is easy to see that

$$E\left(e^{-\eta W_g(t)}\,\middle|F(t),\, Z_0 = i\right) = \prod_{\alpha,\beta=1}^{m} \xi_{\alpha\beta}^{f_{\alpha\beta}(t)}(\eta) \qquad (5.2)$$

and so,

$$E\left(e^{-\eta W_g(t)}\,\middle|Z_0 = i\right) = E\left\{\prod_{\alpha,\beta=1}^{m} \xi_{\alpha\beta}^{f_{\alpha\beta}(t)}(\eta)\,|Z_0 = i\right\}$$

But the right hand side is nothing but the generating function of the probability distribution of $F(t)$, conditional on $Z_0 = i$, with $\xi_{\alpha\beta}(\eta)$ as the arguments, and so by (2.16), its L. - S.T. is the i^{th} element of the column vector

$$\{I - q(s) \,\Box\, \xi(\eta)\}^{-1}(I - q(s))\underline{e}\ , \qquad (5.3)$$

where $\xi(\eta) = [\xi_{ij}(\eta)]$. (5.3) therefore gives the Laplace transform of the distribution of $W_g(t)$. One can obtain the moments of $W_g(t)$ by differentiating (5.3) w.r.t. η and then expanding in powers of s . Alternatively, the mean and variance can also be derived as below:

$$E\left(W_g(t)\,|Z_0 = i\right) = E\left\{E\left(W_g(t)\,|F(t),\ Z_0 = i\right)\right\}$$

$$= E\left\{\sum_{\alpha,\beta=1}^{m} f_{\alpha\beta}(t)\,\frac{A_{\alpha\beta}}{p_{\alpha\beta}}\ \Big|\ Z_0 = i\right\}$$

$$= \sum_{\alpha,\beta=1}^{m} E\left(f_{\alpha\beta}(t)\,|Z_0 = i\right)\cdot\frac{A_{\alpha\beta}}{p_{\alpha\beta}}\ , \tag{5.4}$$

where

$$A_{\alpha\beta} = \int_0^\infty g(\alpha,\beta,x)\,dQ_{\alpha\beta}(x)\ . \tag{5.5}$$

Let further

$$B_{\alpha\beta} = \int_0^\infty g^2(\alpha,\beta,x)\,dQ_{\alpha\beta}(x)\ . \tag{5.6}$$

and then,

$$V\left(W_g(t)\,|Z_0 = i\right) = E\left\{V\left(W_g(t)\,|F(t),\ Z_0 = i\right)\right\} + V\left\{E\left(W_g(t)\,|F(t),\ Z_0 = i\right)\right\}$$

$$= \sum_{\alpha,\beta=1}^{m} E\left(f_{\alpha\beta}(t)\,|Z_0 = i\right)\left(\frac{B_{\alpha\beta}}{p_{\alpha\beta}} - \frac{A^2_{\alpha\beta}}{p^2_{\alpha\beta}}\right) \tag{5.7}$$

$$+ \sum_{\alpha,\beta=1}^{m}\ \sum_{\gamma,\delta=1}^{m} \mathrm{Cov}\left(f_{\alpha\beta}(t),\ f_{\gamma\delta}(t)\,|Z_0 = i\right)\frac{A_{\alpha\beta}A_{\gamma\delta}}{p_{\alpha\beta}p_{\gamma\delta}}\ .$$

We set

$$\sum_{\beta=1}^{m} A_{\alpha\beta} = A_\alpha \quad \text{and} \quad \sum_{\beta=1}^{m} B_{\alpha\beta} = B_\alpha\ . \tag{5.8}$$

By substituting for $E\left(f_{\alpha\beta}(t)\,|Z_0 = i\right)$ and $\mathrm{Cov}\left(f_{\alpha\beta}(t),\ f_{\gamma\delta}(t)\,|Z_0 = i\right)$, from (4.12) and (4.13), one can readily show that

$$\frac{1}{t}\,E\left(W_g(t)\,|Z_0 = i\right) \sim \sum_{\alpha=1}^{m} \frac{d_\alpha A_\alpha}{k_1}\ , \tag{5.9}$$

and

$$\frac{1}{t} V\left(W_g(t) \mid Z_0 = i\right) \sim \sum_{\alpha=1}^{m} \frac{B_\alpha d_\alpha}{k_1} - 2 \sum_{\alpha=1}^{m} \frac{A_\alpha d_\alpha}{k_1} \sum_{\gamma,\delta=1}^{m} \frac{A_{\gamma\delta} d_\gamma c_{\gamma\delta}}{p_{\gamma\delta} k_1}$$

$$+ 2ak_1 \left(\sum_{\alpha=1}^{m} \frac{A_\alpha d_\alpha}{k_1} \right)^2 + 2 \sum_{\alpha,\gamma,\delta=1}^{m} \frac{A_\alpha A_{\gamma\delta} d_\gamma h_{\delta\alpha}}{k_1^2} \qquad (5.10)$$

If

$$W_R(t) = \sum_{n=1}^{N(t)} R(J_{n-1}, J_n, X_n) \qquad (5.11)$$

is another cumulative process, associated with the M.R.P. and if

$$A_{\alpha\beta}^{(R)} = \int_0^\infty R(\alpha,\beta,x)\, dQ_{\alpha\beta}(x) \quad , \qquad (5.12)$$

$$B_{\alpha\beta}^{(R)} = \int_0^\infty R^2(\alpha,\beta,x)\, dQ_{\alpha\beta}(x) \quad , \qquad (5.13)$$

$$K_{\alpha\beta} = \int_0^\infty g(\alpha,\beta,x) R(\alpha,\beta,x)\, dQ_{\alpha\beta}(x) \quad , \quad \sum_\beta K_{\alpha\beta} = K_\alpha \qquad (5.14)$$

it can be shown, in the same way as for $V\, W_g(t) \mid Z_0 = i$, that

$$\frac{1}{t} \operatorname{Cov}\left(W_g(t),\, W_R(t) \mid Z_0 = i\right) \sim \sum_{\alpha=1}^{m} \frac{K_\alpha d_\alpha}{k_1} - \sum_{\alpha=1}^{m} \frac{A_\alpha d_\alpha}{k_1} \sum_{\gamma,\delta=1}^{m} \frac{A_{\gamma\delta}^{(R)} d_\gamma c_{\gamma\delta}}{p_{\gamma\delta} k_1}$$

$$- \sum_{\alpha=1}^{m} \frac{A_\alpha^{(R)} d_\alpha}{k_1} \sum_{\gamma,\delta=1}^{m} \frac{A_{\gamma\delta} d_\gamma c_{\gamma\delta}}{p_{\gamma\delta} k_1} + 2ak_1 \sum_{\alpha=1}^{m} \left(\frac{A_\alpha d_\alpha}{k_1} \right)$$

$$\times \sum_{\alpha=1}^{m} \left(\frac{A_\alpha^{(R)} d_\alpha}{k_1} \right) + \sum_{\alpha,\gamma,\delta=1}^{m} \frac{d_\gamma h_{\delta\alpha}}{k_1^2} \left(A_\alpha A_{\gamma\delta}^{(R)} + A_\alpha^{(R)} A_{\gamma\delta}\right)$$

$$(5.15)$$

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