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A PARAMETRIC FORM FOR Θ IN A COMBINED STATISTIC OF THE FORM $P_1 P_2^{\Theta}$

by

Elinor S. Pape

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A PARAMETRIC FORM FOR Θ IN A COMBINED STATISTIC OF THE FORM $P_1 P_2^{\Theta}$

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Elinor Sandborn Pape
(B.S., Southern Methodist University, 1960)

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Pape, Elinor Sandborn

B.S., Southern Methodist University, 1960

A Parametric Form for Θ in a Combined Statistic of the Form $P_1 P_2^{\Theta}$

Adviser: Associate Professor John T. Webster

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Consider two independent test statistics P_1 and P_2^{Θ} ,

$$P_i = \int_{S_i}^{\infty} k_i(s|\lambda_i = 1)\, ds ,$$

where k_i is the conditional density of S_i , and $S_i/\lambda_i^{-1} \sim f(m_i , rm_i , 0)$, $i = 1, 2$. The problem considered in this thesis is how to combine the information from these two statistics when testing the null hypothesis $\lambda_i = 1$. This paper recommends the use of the combined statistic $P_1 P_2^{(m_2/m_1)\cdot .43(1-\lambda_2)/(1-\lambda_1)}$ and discusses when the ratio $1-\lambda_2/1-\lambda_1$ can be considered to be greatly biased as an estimator of the function of λ_1 and λ_2 which would maximize the power of the test.

This paper further advocates, for more than two independent tests, the combined statistic

$$\prod_{i=1}^{n} P_i^{(m_i)\cdot .43(1-\lambda_i)}$$

maintaining that these exponents approximate pairwise optimal weighting of the individual tests.

The utility of these combined statistics arises from a statistical model with random treatments, blocks, and interaction effects where subgroups have different numbers of readings. Instead of producing one F-statistic for testing treatment effects, the test data actually produces several independent F-statistics.

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CHAPTER I

INTRODUCTION

The general problem under consideration is that of combining test statistics, S_1 and S_2 , from two independent tests into one statistic of the form $P_1 P_2^{\Theta}$ for the purpose of testing a particular null hypothesis, H_0 . To define P_i let the general density of S_i be $g_i(x)$, and the conditional density under H_0 of S_i be $k_i(x|H_0)$. Then $P_i = \int_{S_i}^{\infty} k_i(x|H_0)\, dx$ and the density of P_i will be referred to as $f_i(x)$. A functional form is sought for the weighting factor Θ in terms of the parameters of the two tests.

This approach to the formulation of a combined statistic can be thought of as a refinement of the method of Fisher [1] in which Θ would be equal to 1, i.e., the combined statistic would be $P_1 P_2$. The use of weighting factor allows us to take into account that the different P's may be based on different amounts of information.

Zelen and Joel [2] considered the problem of finding an exponential weighting factor for the incomplete block design with fixed treatment effects where random block effects yield intra-block information which is independent of inter-block information. The two statistics which result in that case are central F's under the null hypothesis and noncentral F's under the alternate hypothesis.

In this paper two independent tests statistics S_1 and S_2 are considered which have the property that the ratios S_1/λ_1^{-1} and S_2/λ_2^{-1} have central F-distributions with respectively (m_1 , n_1) and (m_2 , n_2) degrees of freedom.

The F-distributions are used to test the null hypothesis that $\lambda_1 = \lambda_2 = 1$. The parameters λ_1 , $\lambda_2 (0 < \lambda_i \leq 1, i = 1, 2)$ are measures of the power of the individual tests and are functions of the variances of the elements of the model.

In an effort to keep this work general, the variances will not be introduced until after a Θ-form is determined in terms of m_1 , n_1 , m_2 , n_2 , λ_1 , and λ_2 . Generality will be sacrificed to the extent of considering only pairs of test statistics in which the ratio of degrees of freedom is identical, i.e., $m_1/n_1 = m_2/n_2 = r^{-1}$. This restriction is made to fit the conditions of the specific problem presented in Chapter IV of this paper. Thus we seek a $\hat{\Theta} = \phi(\lambda_1 , \lambda_2 , m_1 , r)$ which has the optimal property of maximizing the power of the combined test resulting from any m_1 , m_2 and r over a wide range of anticipated λ_1 and λ_2 values.

CHAPTER II

POWER DETERMINATION FOR THE COMBINED STATISTIC $P_1P_2^{\Theta}$

The critical points of the $P_1P_2^{\Theta}$ statistic are tabled for $\alpha = .05$ and .01 by Zelen [3] for Θ between 0 and 1 in increments of .1 . They can be determined from the equations

for $0 < C < 1$

$$\alpha = \text{Prob}(P_1P_2^{\Theta} \leq C|H_0) = \begin{cases} C & \text{for } \Theta = 0 \\ \dfrac{C - \Theta C^{1/\Theta}}{1 - \Theta} & \text{for } 0 < \Theta < 1 \\ C(1 - \ln C) & \text{for } \Theta = 1 \end{cases}$$

which are derived from the uniform distribution property of P_1 and P_2 under the null hypothesis.

The power of the combined test then is

$$\text{Power} = \text{Prob}(P_1P_2^{\Theta} \leq C_\alpha|\lambda_1 , \lambda_2) = \iint_{\omega} f_1(x|\lambda_1)\, f_2(y|\lambda_2)\, dxdy \tag{1}$$

where ω is the critical region determined under the null hypothesis and f_1 and f_2 are the density functions of P_1 and P_2 , respectively.

Although the critical region is easily found in P_1 , P_2 space, $(P_1P_2^{\Theta} \leq C_\alpha)$, the density of P_1 gives difficulty. By definition

$$P_1 = \int_{S_1}^{\infty} k_1(t|\lambda_1 = 1)\, dt = \int_{S_1/\lambda_1^{-1}}^{\infty} g_1(t)\, dt$$

Note that when $\lambda_1 = 1$, S_1 has a central F-distribution. Because the Beta distribution yields a slightly simpler form (and because the Beta distribution is used extensively later in the problem) we make the transformation

$$X_1 = \frac{m_1 S_1}{n_1 + m_1 S_1}$$

or equivalently

$$S_1 = \frac{n_1 X_1}{m_1 (1 - X_1)}$$

where $X_1 \sim \text{Beta}\ (m_1/2\ ,\ n_1/2)$ when $\lambda_1 = 1$.

To find the density of X_1 when $\lambda_1 \neq 1$ consider new variables X_{a1} and S_{a1}, and $X_{1\lambda}$ and $S_{1\lambda}$, related as X_1 and S_1 . The variable X_{a1} shall be considered to have the Beta distribution and $X_{1\lambda}$ and X_{a1} shall be related through $S_{1\lambda}$ and S_{a1} .

From the relationship

$$S_{a1} = [S_{1\lambda}]\lambda_1$$

where S_{a1} has a central F-distribution we see that

$$S_{a1} = \frac{n_1 X_{a1}}{m_1 (1 - X_{a1})} = \lambda_1 \left[\frac{n_1 X_{1\lambda}}{m_1 (1 - X_{1\lambda})} \right] = \lambda_1 [S_{1\lambda}] \ .$$

Solving for X_{a1} in terms of $X_{1\lambda}$

$$X_{a1} = \frac{\lambda_1 X_{1\lambda}}{1 - X_{1\lambda} + \lambda_1 X_{1\lambda}}$$

and equivalently

$$X_{1\lambda} = \frac{X_{a1}}{X_{a1} + \lambda_1 - X_{a1}\lambda_1} \ .$$

Then $P_1 = \int_{S_{1\lambda}}^{\infty} g_1(t)\,dt = \int_{X_{1\lambda}}^{1} h(x_1 | \lambda_1)\,dx_1 = \int_{X_{a1}}^{1} \text{Beta}(x_1)\,dx_1$.

From the relationship of $X_{1\lambda}$ and X_{a1} we find the density of $X_{1\lambda}$, that is we find the density of X_1 when $\lambda \neq 1$:

$$h(x_1|\lambda_1) = \frac{\lambda_1}{\beta(m_1, n_1)} (\lambda_1 x_1)^{\frac{m_1}{2}-1} (1-x_1)^{\frac{m_1}{2}-1} [1-x_1+\lambda_1 x_1]^{\frac{-m_1+n_1}{2}}$$

Under the null hypothesis that $\lambda_1 = 1$ the above expression becomes the density of a Beta distribution. It is through this density that we get an expression for the density of P_1 . From the properties of the probability integral transformation [4] we find:

$$f_1(p|\lambda_1) = f_1(p|H_a) = \frac{h(x_1|H_a)}{h(x_1|H_0)} = \frac{h(x_1|\lambda_1)}{h(x_1|1)}$$

where $x_1 = g(p_1)$ and $g(p_1)$ means the solution of the equation

$$P_1 = \int_{x_1}^{1} \text{Beta}(t)\,dt = 1 - I_{x_1}\left(\frac{m_1}{2}, \frac{n_1}{2}\right)$$

for x_1 .

From $h(x_1|\lambda_1)$ and from $h(x_1|1) = \frac{1}{\beta(m,n)} x_1^{\frac{m_1}{2}-1} (1 - x_1)^{\frac{n_1}{2}-1}$

we find:

$$f_1(p_1|\lambda_1) = \lambda_1^{\frac{m_1}{2}} \left[1 - x_1 + \lambda_1 x_1\right]^{-\left(\frac{m_1+n_1}{2}\right)}$$

where $x_1 = g(p_1)$.

Since the density of P_1 is not an explicit function of P_1 , numerical integration of the power function was used. For that purpose the form (1) of the power function could be used but for simplicity consider a second integral transformation

$$\pi_i = 1 - \int_0^{P_i} f_i(p|\lambda_1)\,dp \qquad i = 1, 2$$

$$\text{Power} = \text{Prob}(\omega|\lambda_1 , \lambda_2) = \iint_\omega d\pi_1 d\pi_2 \tag{2}$$

The mapping of the critical region from P_1 , P_2 space into π_1 , π_2 space is accomplished through the incomplete Beta function realizing that

$$1 - \int_0^{P_1} f_1(p|\lambda_1)\,dp = \int_{\frac{\lambda_1 x_1}{1-x_1+\lambda_1 x_1} = x_{a1}}^{1} \text{Beta}(X_1)\,dx_1 = \pi_1$$

The particular mapping procedure used selected π_1 values and solved for the corresponding π_2 coordinate on the boundary of the critical region in the following fashion

<u>Step 1.</u> Select π_1

<u>Step 2.</u> Find X_{a1} from the equation $\pi_1 = 1 - I_{X_{a1}}\left(\frac{m_1}{2} , \frac{n_1}{2}\right)$

<u>Step 3.</u> Find X_1 from the equation $X_1 = \dfrac{X_{a1}}{\lambda_1 - \lambda_1 X_{a1} + X_{a1}}$

<u>Step 4.</u> Find P_1 from the equation $P_1 = 1 - I_{X_1}\left(\frac{m_1}{2} , \frac{n_1}{2}\right)$

<u>Step 5.</u> Find P_2 from the equation $P_1 P_2^{\Theta} = C_\alpha$

<u>Step 6.</u> Find X_2 from the equation $P_2 = 1 - I_{X_2}\left(\frac{m_2}{2} , \frac{n_2}{2}\right)$

<u>Step 7.</u> Find X_{2a} from the equation $X_{2a} = \dfrac{\lambda_2 X_2}{1 - X_2 + \lambda_2 X_2}$

<u>Step 8.</u> Find π_2 from the equation $\pi_2 = 1 - I_{X_{2a}}\left(\dfrac{m_2}{2}, \dfrac{n_2}{2}\right)$

The value of the power,

$$\iint_{\omega} f(P_1|\lambda_1) f(P_2|\lambda_2) dP_1 dP_2 \quad ,$$

is approximated through the form (2) by calculating the area under the set of points thus generated.

The technique used for finding $I_X\left(\frac{m}{2}, \frac{n}{2}\right)$ in this study was to read incomplete Beta tables (K. Pearson [5]) into the computer and interpolate. The power figures calculated were considered to be accurate to the third decimal place.

CHAPTER III

FORMULATION OF $\hat{\Theta}$

The values of Θ(or $1/\Theta$) were varied from 0 to 1 in increments of .1 . Note that

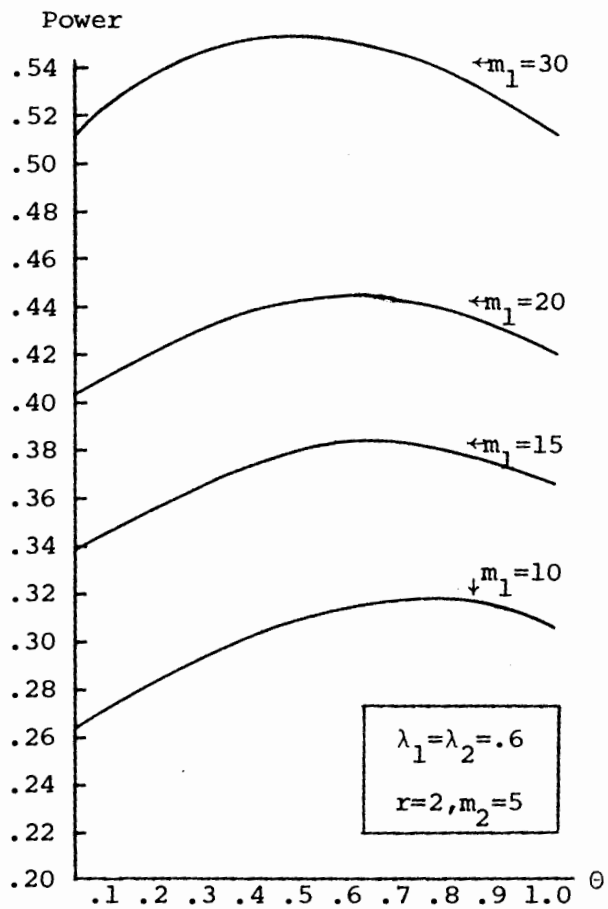
$$\text{Prob}(P_1 P_2^{\Theta} < C_{\alpha,\Theta}) = \text{Prob}\left[P_2 P_1^{\frac{1}{\Theta}} < (C_{\alpha,\Theta})^{\frac{1}{\Theta}}\right] = \text{Prob}\left[P_2 P_1^{\frac{1}{\Theta}} < C_{\alpha,1/\Theta}\right] .$$

In this manner power can be examined over a full range of Θ values from 0 to $+\infty$ for any specific values of m_1 , m_2 , r , λ_1 , and λ_2 .

The power vs. Θ curves thus obtained are unimodal and their crests are quite flat; e.g. (Figure 1). From these curves Θ_p (the value of which optimizes power) can be estimated. It is for these optimizing values of Θ that an estimator is desired. The flatness of the curve crests indicates that an estimator of Θ_p need not be precise to be of value.

Fixing $\lambda_1 = \lambda_2 = \lambda$, the parameters m_1 , m_2 , and r were varied to produce the results of Table I. The variations with λ are not presented because they were considered beyond the precision of the power calculation technique. The nature of the variation with λ is in the direction of 1 as λ is decreased. The value of Θ_p when $\lambda = .2$ may exceed Θ_p when $\lambda = .8$ by at most .1 , where m_1 and m_2 are ordered so that Θ_p is less than 1 . The figures in Table I represent an average over the various λ values in the cases where the variation appeared.

The noticeable similarity in the Θ_p values when the ratio of m_1 to



Illustrative Curves

Power vs. Θ

FIGURE 1.

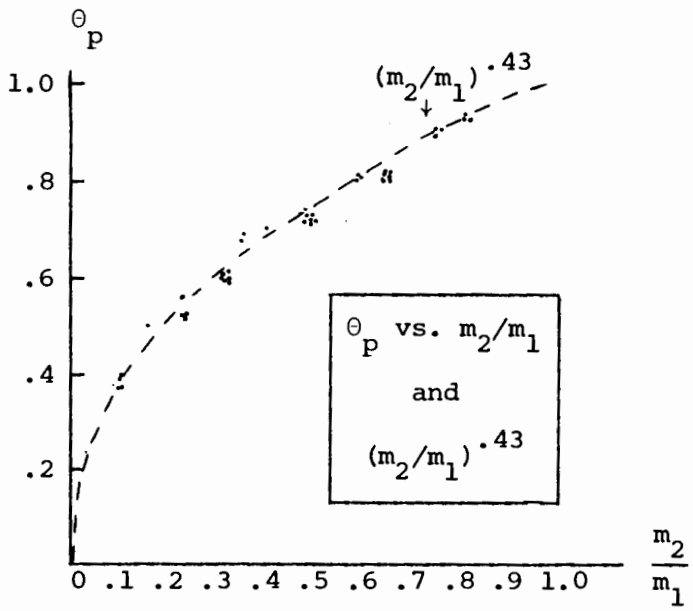


FIGURE 2.

TABLE I

Θ_p Values With $\lambda_1 = \lambda_2$

r	m_1	m_2	m_2/m_1	Θ_p	r	m_1	m_2	m_2/m_1	Θ_p	r	m_1	m_2	m_2/m_1	Θ_p
1	10	5	.5	.7	2	10	5	.5	.7	2	20	15	.75	.9
1	15	5	.33	.6	2	15	5	.33	.6	2	30	15	.50	.7
1	20	5	.25	.5	2	20	5	.25	.5	2	40	15	.38	.7
1	15	10	.67	.8	2	30	5	.17	.5	2	50	15	.30	.6
1	20	10	.50	.7	2	40	5	.12	.4	2	30	20	.67	.8
1	20	15	.75	.9	2	50	5	.10	.4	2	40	20	.50	.7
					2	15	10	.67	.8	2	50	20	.40	.7
5	6	3	.5	.75	2	20	10	.50	.7	2	40	30	.75	.9
5	10	6	.6	.8	2	30	10	.33	.6	2	50	30	.6	.8
					2	40	10	.25	.6	2	50	40	.8	.9
					2	50	10	.20	.5					

m_2 is constant, irrespective of r value, lead to the plotting of Figure 2. Fitting these points with an equation of the form $\Theta_p = (m_2/m_1)^K(e)$ or $\ln \Theta_p = K\, n(m_2/m_1) + e'$ where e and e' represent errors leads to a least squares estimate of K of .43. That is, $\hat{\Theta}_p = (m_2/m_1)^{.43}$ when $\lambda_1 = \lambda_2$.

The results of varying λ_1 and λ_2 appear in Table II. Without loss of generality λ_1 has been restricted to values less than λ_2 . That is, the test with the smaller (more powerful) λ parameter is assigned the subscript 1 . However if $m_2 > m_1$ the second test could actually be the more powerful. Beside the values of Θ_p appear the values Θ_p(adj.) = $(m_1/m_2)^{.43}\Theta_p$. The similarity of these adjusted values for any fixed values of λ_1 and λ_2 leads to the proposed prediction form $\hat{\Theta}_p = (m_2/m_1)^{.43}$ $g(\lambda_1 , \lambda_2)$. A simple algebraic form for the function $g(\lambda_1 , \lambda_2)$ that explains all of its variational characteristics has not been deduced. Therfore, the best estimate of $g(\lambda_1 , \lambda_2)$, $f(\lambda_1 , \lambda_2)$, can be read from Table III, entering the chart with the values of $1-\lambda_1$ and $1-\lambda_2$. The values of $f(\lambda_1 , \lambda_2)$ presented in the chart were obtained by averaging the adjusted figures from Table II and additional computer calculations run with $m_1 = m_2$.

Although a simple algebraic form is lacking that fully explains the function, the form $(1-\lambda_2)/(1-\lambda_1)$ is good over a large segment of what might be called λ_1 , λ_2 space. Also this simple λ-ratio is never very far wrong. The dotted lines on the chart of Table III delineate the contours that would be generated by the λ-ratio. To the extent that ".k" values of $f(\lambda_1 , \lambda_2)$ fall in the fan-shaped contour that runs from the point (0,0) to (1, .k) for k = 1, 2, 3 ··· 9 , this ratio is a good predictor of $f(\lambda_1 , \lambda_2)$. In general the ratio $(1-\lambda_2)/(1-\lambda_1)$ appears to be high when power is high and low when power is low. Its marked systematic variations

r=2

$\frac{1-\lambda_2}{1-\lambda_1}$	λ_1	λ_2	$m_1=m_2=5$, $m_1=m_2=10$ Θ_p	Θ_p(adj)	$m_1=m_2=15$, $m_1=m_2=20$ Θ_p	Θ_p(adj)	$m_1=5$ $m_2=10$ Θ_p	Θ_p(adj)	$m_1=10$ $m_2=5$ Θ_p	Θ_p(adj)
1.0	λ	λ	1.0	1.0	1.0	1.0	1/.7	1.05	.7	.95
.75	.2	.4	$.6^+$	$.6^+$	.7	.7	.8	.6	.5	.67
.67	.4	.6	.6	.6	.65	.65	.8	.6	.45	.6
.5	.6	.8	.45	.45	.5	.5	.65	.5	.35	.45
.5	.2	.6	.4	.4	.5	.5	.55	.4	.33	.45
.3	.4	.8	.3	.3	.33	.33	.4	.3	.25	.35
.25	.2	.8	.2	.2	.25	.25	.3	.2	.15	.2

r=2

$\frac{1-\lambda_2}{1-\lambda_1}$	λ_1	λ_2	$m_1=10$ $m_2=30$, $m_1=5$ $m_2=15$ Θ_p	Θ_p(adj)	$m_1=30$ $m_2=10$, $m_1=15$ $m_2=5$ Θ_p	Θ_p(adj)	$m_1=5$ $m_2=30$ Θ_p	Θ_p(adj)	$m_1=30$ $m_2=5$ Θ_p	Θ_p(adj)
1.0	λ	λ	1/.6	1.05	.6	.95	1/.5	.9	.5	1.1
.75	.2	.4	1.0	.6	.45	.7	1/.7	.65	.4	.85
.67	.4	.6	1.0	.6	.4	.65	1/.75	.6	.3	.65
.5	.6	.8	.8	.5	.3	.5	1.0	.45	.25	.55
.5	.2	.6	.6	.35	.3	.5	$.8^+$	.4	.25	.55
.3	.4	.8	.5	.3	.2	.3	$.6^+$	.3	.2	.4
.25	.2	.8	.3	.2	.1	.15	.4	.2	.1	.2

TABLE II--*Continued*

r=2

$\frac{1-\lambda_2}{1-\lambda_1}$	λ_1	λ_2	$m_1=5$ $m_2=50$ Θ_p	Θ_p(adj)	$m_1=10$ $m_2=50$ Θ_p	Θ_p(adj)	$m_1=10$ $m_2=15$ Θ_p	Θ_p(adj)	$m_1=15$ $m_2=10$ Θ_p	Θ_p(adj)
1.0	λ	λ	1/.4	.95	1/.5	1.0	1/.8	1.05	.8	.95
.75	.2	.4	1/.6	.6	1/.75	.67	.8	.67	.6	.7
.67	.4	.6	1/.6	.6	1/.9	.55	.7	.6	.5	.6
.5	.6	.8	1/.75	.5	1.0	.5	.55	.45	.4	.5
.5	.2	.6	1/.95	.4	$.8^-$	.4	.5	.4	.4	.5
.3	.4	.8	$.8^-$	.3	$.6^-$	.3	$.4^-$	.3	.3	.35
.25	.2	.8	.5	.2	.33	.15	.3	.25	.2	.25

r=2

$\frac{1-\lambda_2}{1-\lambda_1}$	λ_1	λ_2	$m_1=10$ $m_2=20$ Θ_p	Θ_p(adj)	$m_1=20$ $m_2=10$ Θ_p	Θ_p(adj)	$m_1=15$ $m_2=20$ Θ_p	Θ_p(adj)	$m_1=20$ $m_2=15$ Θ_p	Θ_p(adj)
1.0	λ	λ	$1/.7^+$	1.0	$.7^+$	1.0	1/.9	1.0	.9	1.0
.75	.2	.4	.9	.67	.5	.67	$.8^-$	.7	.65	.75
.67	.4	.6	$.8^+$	.6	.5	.67	$.6^-$	.5	.55	.6
.5	.6	.8	.65	.5	.4	.55	.55	.5	.45	.5
.5	.2	.6	.55	.4	.35	.45	.5	.45	.4	.45
.3	.4	.8	.4	.3	.25	.35	$.4^-$	.35	.3	.35
.25	.2	.8	.3	.2	.2	.3	.25	.2	.2	.2

TABLE II--*Continued*

r=2 $\frac{1-\lambda_2}{1-\lambda_1}$	λ_1	λ_2	$m_1=15$ $m_2=30$ Θ_p	Θ_p(adj)	$m_1=30$ $m_2=15$ Θ_p	Θ_p(adj)	$m_1=15$ $m_2=50$ Θ_p	Θ_p(adj)	$m_1=50$ $m_2=15$ Θ_p	Θ_p(adj)
1.0	λ	λ	1/.7	1.05	.7	.95	1/.6	1.0	.6	1.0
.75	.2	.4	.9	.67	.5	.67	1.0	.6	.4	.67
.67	.4	.6	$.8^+$	.6	.5	.67	1.0	.6	.4	.67
.5	.6	.8	.65	.5	.4	.55	.8	.5	.3	.5
.5	.2	.6	.55	.4	.35	.45	.7	.4	.3	.5
.3	.4	.8	.4	.3	.25	.35	$.5^+$	.3	.2	.35
.25	.2	.8	.3	.2	.2	.25	.4	.25	---	---

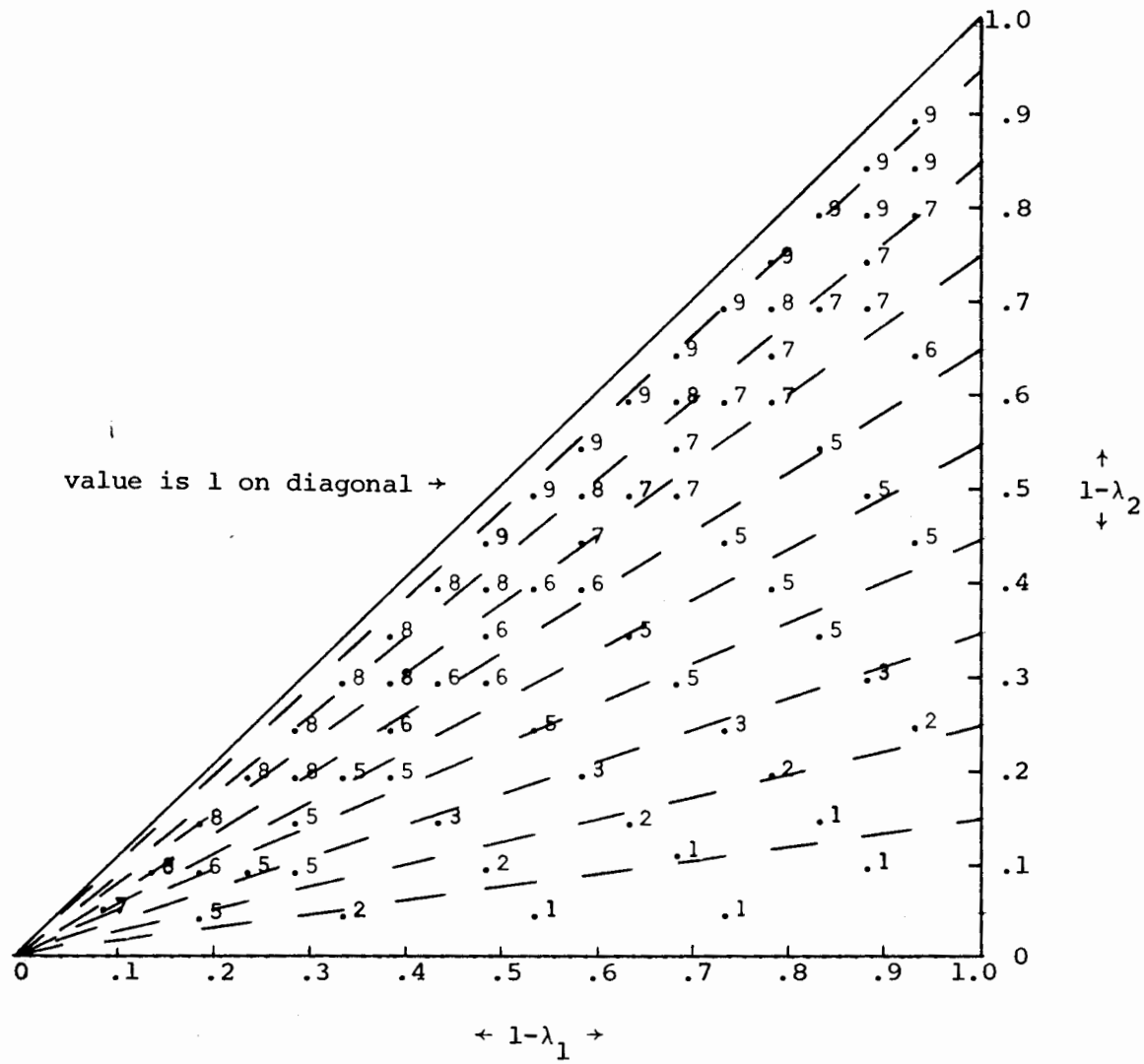
r=2 $\frac{1-\lambda_2}{1-\lambda_1}$	λ_1	λ_2	$m_1=20$ $m_2=30$ Θ_p	Θ_p(adj)	$m_1=30$ $m_2=20$ Θ_p	Θ_p(adj)	$m_1=20$ $m_2=50$ Θ_p	Θ_p(adj)	$m_1=50$ $m_2=20$ Θ_p	Θ_p(adj)
1.0	λ	λ	1/.8	1.05	.8	.95	1/.7	.95	.7	1.05
.75	.2	.4	----	----	---	---	1.0	.67	.5	.75
.67	.4	.6	$.8^-$	.65	.55	.65	.9	.6	.45	.65
.5	.6	.8	$.6^-$	.5	$.4^+$	.5	.7	.45	.35	.5
.5	.2	.6	.55	.45	.4	.5	.65	.45	.3	.45
.3	.4	.8	.4	.35	.3	.35	.5	.35	.25	.35
.25	.2	.8	.25	.2	.2	.25	.35	.25	.1	.15

TABLE II--*Continued*

r=5

$\frac{1-\lambda_2}{1-\lambda_1}$	λ_1	λ_2	$m_1=m_2=3, 6$		$m_1=m_2=10$		$m_1=3$ $m_2=6$		$m_1=6$ $m_2=3$	
			Θ_p	Θ_p(adj)	Θ_p	Θ_p(adj)	Θ_p	Θ_p(adj)	Θ_p	Θ_p(adj)
1.0	λ	λ	1.0	1.0	1.0	1.0	1/.75	1.0	.75	1.0
.78	.1	.3	.7	.7	.7	.7	.9	.67	.55	.7
.7	.3	.5	.67	.67	.7	.7	.9	.67	.5	.7
.6	.5	.7	.55	.55	$.6^-$	.6	.75	.55	.5	.7
.5	.7	.9	.5	.5	$.5^+$	.5	.5	.37	.25	.35
.43	.3	.7	.5	.5	.5	.5	.5	.35	.3	.4
.25	.5	.9	.2	.2	.2	.2	.25	$.2^-$	.16	.2
.22	.1	.7	.25	.25	.3	.3	.3	$.2^+$	.25	.3
.14	.3	.9	.14	.14	.15	.15	.2	.15	.1	.15
.11	.1	.9	.1	.1	.14	.14	.1	.1	.1	.15

TABLE II--*Continued*



Values of $f(\lambda_1 , \lambda_2)$

TABLE III

from true values makes it undesirable as a predictor, but its simplicity coupled with the knowledge that precision is not critical for this problem make the ratio very appealing.

Hence the proposed predictor of Θ_p , the value of Θ which maximizes power, is $(m_2/m_1)^{.43} f(\lambda_1 , \lambda_2)$ where $f(\lambda_1 , \lambda_2)$ is read from Table III or approximated by the ratio $(1-\lambda_2)/(1-\lambda_1)$.

In both cases $\hat{\Theta}_p$ is a function of λ_1 and λ_2 . That is, one must select values of λ_1 and λ_2 for which he wishes to maximize power. The specific tests from which the λ_1 and λ_2 parameters arise should be studied to aid in making a choice of these parameters. However, even in the case where no knowledge of λ_1 and λ_2 is available, the use of $\hat{\Theta}_p = (m_2/m_1)^{.43}$ (which is equivalent to assuming $\lambda_1 = \lambda_2$) would be more apt to maximize power than the choice $\Theta = 1$.

Consideration of a random model with interaction and unequal numbers in the subgroups will furnish an example for the problem of determining values for λ_1 and λ_2 .

CHAPTER IV

THE RANDOM MODEL WITH INTERACTION AND NUMBER OF READINGS IN SUBGROUPS AT TWO DIFFERING LEVELS

Consider the model $y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$

$$i = 1, 2, 3, \cdots, a$$

$$j = 1, 2, 3, \cdots, b$$

$$k = 1, 2, 3, \cdots, n_{ij}$$

where μ is an unknown constant and α_i , β_j , $(\alpha\beta)_{ij}$, and ε_{ijk} are independent random variables from normal populations with zero means and variances σ^2_α , σ^2_β , $\sigma^2_{\alpha\beta}$, and σ^2 respectively.

Webster [6] considers the conditions under which a single approximate F-statistic provides a good test for the hypothesis of no treatment effects. Webster points out in the same paper that even when these conditions are not met but n_{ij} takes on only two values the results yield three independent F-statistics testing the same hypothesis, that of no treatment effects.

That is,

$$n_{ij} = n_1 \quad \text{for } a_1 \text{ of the levels of } \alpha$$

$$= n_2 \quad \text{for } a_2 \text{ of the levels of } \alpha \text{ , } a_1 + a_2 = a \text{ .}$$

Consider $n_1 > n_2$ with no loss of generality.

Let SSA_1 , and $SSAB_1$ be the sums of squares for α and $(\alpha\beta)$ from the first group, using cell means.

Let SSA_2 and $SSAB_2$ be the sums of squares for α and $(\alpha\beta)$ from the second group, using cell means.

Let SSA and SSAB be the sums of squares for α and $(\alpha\beta)$ from all of the readings, using cell means.

From Analysis of Variance Table

TREATMENTS			
Group I	a_1-1	SSA_1	$\frac{\sigma^2}{n_1} + \sigma^2_{\alpha\beta} + b\sigma^2_{\alpha}$
Group II	a_2-1	SSA_2	$\frac{\sigma^2}{n_2} + \sigma^2_{\alpha\beta} + b\sigma^2_{\alpha}$
Bonus	1	$SSA-SSA_1-SSA_2$	$\frac{\sigma^2}{a}\left(\frac{a_2}{n_1} + \frac{a_1}{n_2}\right) + \sigma^2_{\alpha\beta} + b\sigma^2_{\alpha}$
TREATMENTS-BLOCKS			
Group I	$(a_1-1)(b-1)$	$SSAB_1$	$\frac{\sigma^2}{n_1} + \sigma^2_{\alpha\beta}$
Group II	$(a_2-1)(b-1)$	$SSAB_2$	$\frac{\sigma^2}{n_2} + \sigma^2_{\alpha\beta}$
Bonus	$(b-1)$	$SSAB-SSAB_1-SSAB_2$	$\frac{\sigma^2}{a}\left(\frac{a_2}{n_1} + \frac{a_1}{n_2}\right) + \sigma^2_{\alpha\beta}$

The three independent statistics to which Webster refers are:

$$\lambda_1 \frac{SSA_1(b-1)}{SSAB_1} \sim F\left[(a_1-1)\ ,\ (a_1-1)(b-1)\right]$$

where

$$\lambda_1 = \frac{\frac{\sigma^2}{n_1} + \sigma^2_{\alpha\beta}}{\frac{\sigma^2}{n_1} + \sigma^2_{\alpha\beta} + b\sigma^2_{\alpha}}$$

$$\lambda_2 \frac{SSA_2(b-1)}{SSAB_2} \sim F\left[(a_2-1)\ ,\ (a_2-1)(b-1)\right]$$

where
$$\lambda_2 = \frac{\frac{\sigma^2}{n_2} + \sigma^2_{\alpha\beta}}{\frac{\sigma^2}{n_2} + \sigma^2_{\alpha\beta} + b\sigma^2_{\alpha}}$$

$$\lambda_3 \frac{(SSA-SSA_1-SSA_2)(b-1)}{(SSAB-SSAB_1-SSAB_2)} \sim F[1\ ,\ b-1]$$

where
$$\lambda_3 = \frac{\frac{\sigma^2}{a}\left[\frac{a_2}{n_1} + \frac{a_1}{n_2}\right] + \sigma^2_{\alpha\beta}}{\frac{\sigma^2}{a}\left[\frac{a_2}{n_1} + \frac{a_1}{n_2}\right] + \sigma^2_{\alpha\beta} + b\sigma^2_{\alpha}}$$

Since the power optimizing technique of formulating a weighting factor developed in this paper is devised for two tests, the first two will be considered. (These two will always have as many degrees of freedom as the third test, and will have more if a_1 and a_2 are greater than 2 .)

The object of this example is to formulate from these two statistics a single statistic of the form $P_1P_2^{\Theta}$, where $P_i = \text{Prob}(F \geq F_i | \sigma^2_{\alpha} = 0)$, $i = 1, 2$.

$$1-\lambda_1 = \frac{b\,\frac{\sigma^2_{\alpha}}{\sigma^2}}{\frac{1}{n_1} + \frac{\sigma^2_{\alpha\beta}}{\sigma^2} + b\,\frac{\sigma^2_{\alpha}}{\sigma^2}} \qquad 1-\lambda_2 = \frac{b\,\frac{\sigma^2_{\alpha}}{\sigma^2}}{\frac{1}{n_2} + \frac{\sigma^2_{\alpha\beta}}{\sigma^2} + b\,\frac{\sigma^2_{\alpha}}{\sigma^2}}$$

The values of b , n_1 , and n_2 are known integers for any set of data. It is necessary only to select the variance ratios $\sigma^2_{\alpha}/\sigma^2$ and $\sigma^2_{\alpha\beta}/\sigma^2$ for which optimum power is desired. To consider the effect on $\hat{\Theta}$ of different selections of variance ratios we look at $(1-\lambda_2)/(1-\lambda_1)$ the λ-ratio approximation to the $f(\lambda_1\ ,\ \lambda_2)$ charted in Table III.

$$\hat{\Theta} \simeq \left(\frac{a_2-1}{a_1-1}\right)^{.43} \frac{1-\lambda_2}{1-\lambda_1}$$

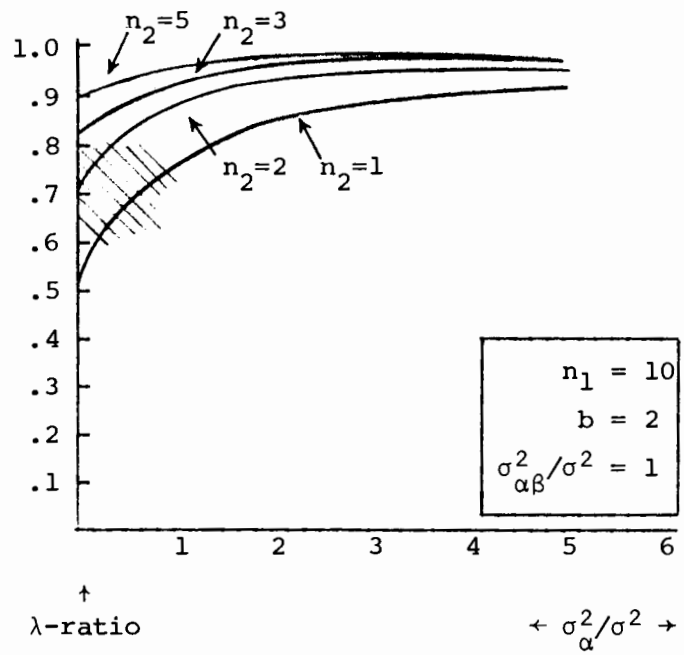
$$\hat{\Theta} \simeq \left(\frac{a_2-1}{a_1-1}\right)^{.43} \frac{\frac{1}{n_1} + \frac{\sigma^2_{\alpha\beta}}{\sigma^2} + b\,\frac{\sigma^2_\alpha}{\sigma^2}}{\frac{1}{n_2} + \frac{\sigma^2_{\alpha\beta}}{\sigma^2} + b\,\frac{\sigma^2_\alpha}{\sigma^2}}$$

From this form it is apparent that if n_1 and n_2 are close to the same value the λ-ratio is effectively unity. However it should be noted that if n_1 is close to n_2 the conditions also hold for use of Webster's approximate F-statistic.

Other conditions that will cause the λ-ratio to be close to unity are large b(number of blocks) or large values of either variance ratio. Even if $1/n_1$ and $1/n_2$ take their extreme values, 0 and 1 respectively, (note we have ordered the tests so that $n_1 > n_2$), the λ-ratio is effectively 1 when $(\sigma^2_{\alpha\beta}/\sigma^2 + b\ \sigma^2_\alpha/\sigma^2) \geqq 20$. And in a more moderate situation where $1/n_1$ and $1/n_2$ take values such as .1 and 1.3 , respectively, the λ-ratio is effectively 1 when $(\sigma^2_{\alpha\beta}/\sigma^2_\alpha + b\ \sigma^2_\alpha/\sigma^2) \geqq 4$. Note that the bias of the λ-ratio estimator of $f(\lambda_1\ ,\ \lambda_2)$ is not apparent at the .95 level of Θ where these computations are made.

Since the model for this problem includes the interaction term, it is not unreasonable to assume some prior knowledge of the $\sigma^2_{\alpha\beta}/\sigma^2$ ratio. If one has knowledge of treatment interaction with blocks but questions the non-zero value of σ^2_α , he might be able to estimate $\sigma^2_{\alpha\beta}/\sigma^2$ and then select a value of σ^2_α/σ^2 at which to optimize power.

Figures 3 and 4 show the variation of the λ-ratio with variation of the σ^2_α/σ^2 ratio for different n-values and b-values. In both graphs the $\sigma^2_{\alpha\beta}/\sigma^2$ ratio is held at a value of 1 . That is, we are considering optimizing the test when $\sigma^2_{\alpha\beta}$ equals σ^2 . A larger value of this ratio will, of course make the λ-ratio values closer to unity. In fact, a

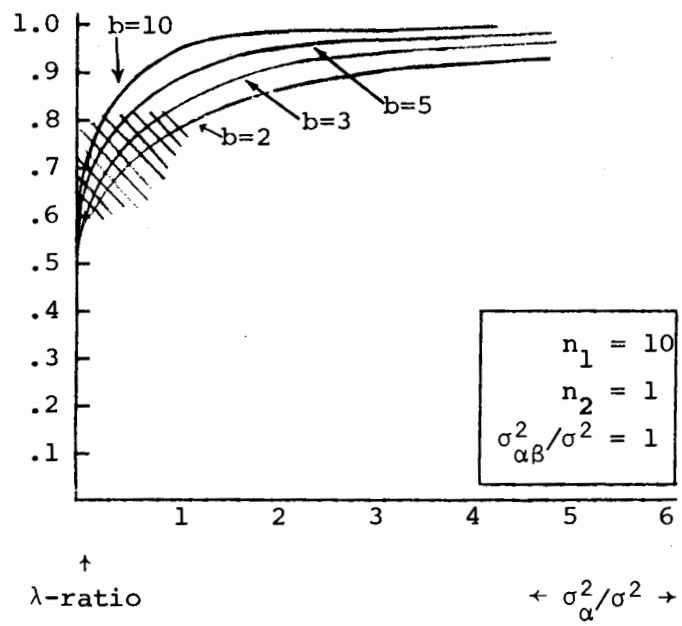


Estimated Interaction

λ-ratio vs. σ^2_α/σ^2

Varying n_1 and n_2

FIGURE 3.



Estimated Interaction

λ-ratio vs. σ^2_α/σ^2

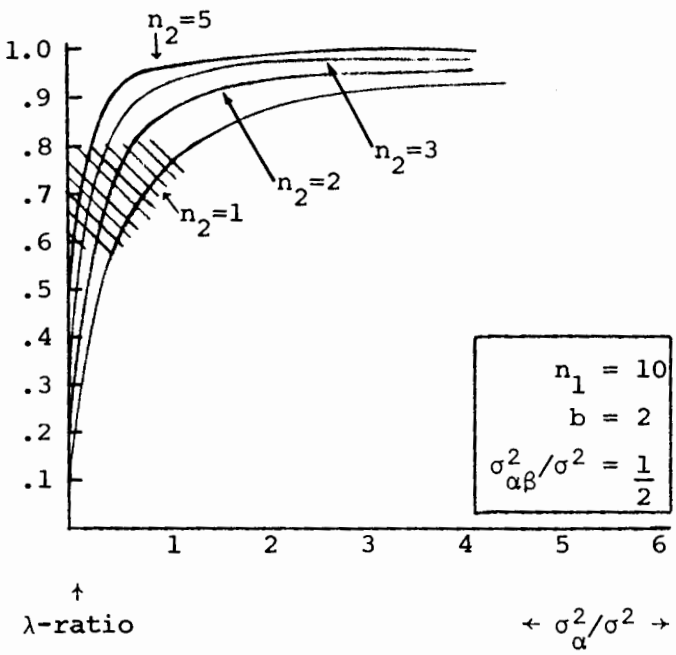
Varying b

FIGURE 4.

value of 2 will raise the intercept value of Figure 4 from .55 to .7 . The cross-hatched area on these two graphs identifies the area in which the λ-ratio values can be considered to be a high estimate of $f(\lambda_1 , \lambda_2)$. (λ-ratio - .15 < $f(\lambda_1 , \lambda_2) \leq \lambda$-ratio).

Another approach to the selection of variance ratios is to first fix the ratio of $\sigma^2_{\alpha\beta}$ to σ^2_α . This approach would be appropriate if one does not have knowledge of interaction magnitude but feels that $\sigma^2_{\alpha\beta}$ will have values in proportion to σ^2_α . The procedure then is to fix the ratio of the two variances for which optimization is desired (1/4, 1/2, 1, etc.) and then select the desired σ^2_α/σ^2 for power optimization.

Figures 5 and 6 show the variation of the λ-ratio with variation of the σ^2_α/σ^2 ratio for different n-values and b-values. In both of these graphs, $\sigma^2_{\alpha\beta}$ is considered to be half of σ^2_α . As in Figures 2 and 3, the area of the graph in which the λ-ratio is a biased estimator of $f(\lambda_1 , \lambda_2)$ is cross-hatched. Again the bias is of the nature: λ-ratio - .15 < < $f(\lambda_1 , \lambda_2) \leq \lambda$-ratio .

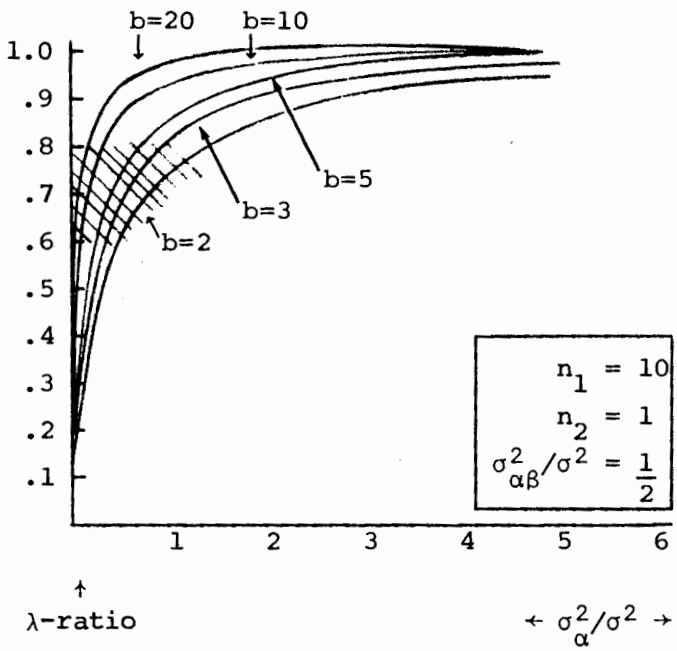


Proportionate Interaction

λ-ratio vs. σ^2_α/σ^2

Varying n_1 and n_2

FIGURE 5.



Proportionate Interaction

λ-ratio vs. σ^2_α/σ^2

Varying b

FIGURE 6.

CHAPTER V

WEIGHTING THREE OR MORE TESTS BY $P_1 P_2^{\Theta_2} P_3^{\Theta_3} \cdots P_n^{\Theta_n}$

The combinatorial technique presented here deals with only two test values, P_1 and P_2 . The question naturally arises as to whether or not the technique can be extended to three or more values. Accurate extension of the technique to three tests ($P_1 P_2^{\Theta_2} P_3^{\Theta_3}$) would require a mapping of the critical region into 3-space so that the numerical calculation of power for any choice Θ_2 and Θ_3 would involve approximation of the volume of a solid. It would then be necessary to generate a power surface over Θ_2 , Θ_3-space from which the power optimizing coordinates of Θ_2 and Θ_3 could be selected.

A simpler approach which is intuitively appealing, if not mathematically defensible, is to optimally combine the third test with each of the first two tests and average the two weights, keeping the first two tests weighted optimally. The purpose of this approach is to make the weighting of each pair of test statistics approximate the optimal weighting of that pair. It is equivalent to using the weighting for each two statistics which will optimize power when the third test is entered into the combined statistic with 0 weight.

It is with regard to pairwise optimality that the λ-ratio set forth in Chapter III becomes particularly useful. The λ-ratio for the statistic $P_1 P_2^{\Theta}$ is the product of the λ-ratios for the statistics $P_1 P_3^{\Theta}$ and $P_3 P_2^{\Theta}$ for

any test P_3 . That is,

$$\frac{1-\lambda_3}{1-\lambda_1} \cdot \frac{1-\lambda_2}{1-\lambda_3} = \frac{1-\lambda_2}{1-\lambda_1}$$

Thus the approximate weighting factor for the third test would be $(m_3/m_1)^{.43}(1-\lambda_3)/(1-\lambda_1)$ where the first test has the weight 1 and the second test has the weight $(m_2/m_1)^{.43}(1-\lambda_2)/(1-\lambda_1)$. Note that multiplying the three weights by the reciprocal of the second or third weighting simply "permutes" the 1 weighting from test one to another test. For instance,

$$\left(P_1 P_2^{\left(\frac{m_2}{m_1}\right)^{.43}\left(\frac{1-\lambda_2}{1-\lambda_1}\right)} P_3^{\left(\frac{m_3}{m_1}\right)^{.43}\left(\frac{1-\lambda_3}{1-\lambda_1}\right)}\right)^{\left(\frac{m_1}{m_2}\right)^{.43}\left(\frac{1-\lambda_1}{1-\lambda_2}\right)}$$

$$= P_2 P_1^{\left(\frac{m_1}{m_2}\right)^{.43}\left(\frac{1-\lambda_1}{1-\lambda_2}\right)} P_3^{\left(\frac{m_3}{m_2}\right)^{.43}\left(\frac{1-\lambda_3}{1-\lambda_2}\right)}$$

Thus using the λ-ratio approximation to $f(\lambda_1 , \lambda_2)$ makes all pairwise weightings optimal with respect to power to the accuracy of that approximation, regardless of the number of tests combined in a multiplicative fashion with exponential weights.

The general form for combining any number , n , of these tests is

$$\prod_{i=1}^{n} P_i^{m_i^{.43}(1-\lambda_i)}$$

The critical values for these statistics can be found from Good's [7] equation: for $0 < q < 1$

$$\text{Prob}\left((P_1^{\Theta_1} P_2^{\Theta_2} P_3^{\Theta_3} \cdots P_n^{\Theta_n}) < q\right) = \sum_{i=1}^{n} \Psi_i \, q^{1/i}$$

where

$$\Psi_i = \theta_i^{n-1}/(\theta_i-\theta_1)(\theta_i-\theta_2)(\cdots)(\theta_i-\theta_{i-1})(\theta_i-\theta_{i+1})\cdots(\theta_i-\theta_n)$$

This equation can be solved for q provided all the θ_i are different.

In the case where n = 3 the following equations can be used to find critical points when two or more of the θ_i are the same, $0 < \theta_i < 4$.

for $0 < c < 1$

$$\text{Prob}(P_1P_2P_3 < c) = c[1 - \ln c + \frac{\ln^2 c}{2}]$$

$$\text{Prob}(P_1P_2P_3^{\theta} < c) = c + \frac{(c - c^{\frac{1}{\theta}})}{1 - \theta} - \frac{c \ln c}{1 - \theta} - \frac{(c - c^{\frac{1}{\theta}})}{(1 - \theta)^2}$$

$$\text{Prob}(P_1P_2^{\theta}P_3^{\theta} < c) = c^{\frac{1}{\theta}} + \frac{c^{\frac{1}{\theta}} \ln c}{1 - \theta} + \frac{c - c^{\frac{1}{\theta}}}{(1 - \theta)^2}$$

Consider several examples. Let P_1 , P_2 , and P_3 be statistics resulting from three independent tests of the same statistical hypothesis which can be characterized by $\lambda_i = 1$, $i = 1, 2, 3$. Let $m_1 = 10$, $m_2 = 5$, and $m_3 = 1$. We want to find the exponents θ_2 and θ_3 which will optimize power of the combined test statistic $P_1P_2^{\theta_2}P_3^{\theta_3}$, when $\lambda_1 = .3$, $\lambda_2 = .5$, and $\lambda_3 = .4$. The proposed form for $\hat{\theta}_2$ is $(5/10)^{.43}(.5/.7) = .5$. The proposed form for $\hat{\theta}_3$ is $(1/10)^{.43}(.6/.7) = .3$. Then for a combined statistic we would use $P_1P_2^{.5}P_3^{.3} = Q$. To find a critical point when $\alpha = 1.05$

$$\begin{aligned}\text{Prob}(Q < q) &= .05\\ &= q/(.5 \cdot .7) - q^2/(.5 \cdot .2) + q^{10/3}/(.7 \cdot .2)\\ &= q/.35 - q^2/.10 + q^{10/3}/.14\\ q &= .018722\end{aligned}$$

Using the parameters of the three independent statistics resulting from the random model with interaction and two levels of subgroup readings (Chapter IV), let $a_1 = 5$, $a_2 = 15$, $n_1 = 10$, $n_2 = 1$, $b = 2$. We seek a combined statistic of the form $P_1^{\theta_1}P_2^{\theta_2}P_3^{\theta_3}$ with optimal power when $\sigma^2_{\alpha\beta}/\sigma^2 = 1$ and $\sigma^2_{\alpha}/\sigma^2 = 1$.

$$\hat{\theta}_2 = \left(\frac{a_2-1}{a_1-1}\right)^{.43}\left(\frac{\frac{1}{n_1} + \frac{\sigma^2_{\alpha\beta}}{\sigma^2} + b\,\frac{\sigma^2_{\alpha}}{\sigma^2}}{\frac{1}{n_2} + \frac{\sigma^2_{\alpha\beta}}{\sigma^2} + b\,\frac{\sigma^2_{\alpha}}{\sigma^2}}\right) = \left(\frac{14}{4}\right)^{.43}\left(\frac{.1 + 1 + 2}{1 + 1 + 2}\right) = 1.34$$

$$\hat{\theta}_3 = \left(\frac{1}{a_1-1}\right)^{.43}\left(\frac{\frac{1}{n_1} + \frac{\sigma^2_{\alpha\beta}}{\sigma^2} + b\,\frac{\sigma^2_{\alpha}}{\sigma^2}}{\frac{a_2}{an_1} + \frac{a_1}{an_2} + \frac{\sigma^2_{\alpha\beta}}{\sigma^2} + b\,\frac{\sigma^2_{\alpha}}{\sigma^2}}\right) = \left(\frac{1}{4}\right)^{.43}\left(\frac{.1 + 1 + 2}{\frac{15}{20\cdot 10} + \frac{5}{20\cdot 1} + 1 + 2}\right) = .51$$

Yielding the combined statistic $P_1P_2^{1.3}P_3^{.5}$, or equivalently $P_2P_1^{.8}P_3^{.4}$.

The above example magnifies the effect the λ-ratio on the weighting factor by having a relatively extreme n_1/n_2 ratio and by having a minimum number of blocks. If instead we use $n_1 = 10$, $n_2 = 5$, $b = 5$, we get the following results:

$$\hat{\theta}_2 = \left(\frac{14}{4}\right)^{.43}\left(\frac{.1 + 1 + 5}{.2 + 1 + 5}\right) = 1.6$$

$$\hat{\theta}_3 = \left(\frac{1}{4}\right)^{.43}\left(\frac{.1 + 1 + 5}{.125 + 1 + 5}\right) = .55$$

Yielding the combined statistic $P_1P_2^{1.6}P_2^{.55}$, or equivalently $P_2P_1^{.6}P_3^{.3}$, which is the same statistic that results when both λ-ratios are entered as 1 .

Thus for the random model problem only extreme values of the n, b, and σ^2 parameters effect the form of the combined statistic. In most

cases the statistic can be formulated using only a_1 and a_2 , the numbers of treatments receiving each of the different number readings.

CHAPTER VI

SUMMARY

The problem considered was to formulate a combined statistic of the $P_1P_2^{\Theta}$-type for two particular F-tests. Mathematical theory was utilized in the solution of this problem only to the point of developing a technique for generating the power value for a particular set of test parameters. From that point on the problem became an excercise in data analysis.

Since the plots of power vs. Θ showed very flat crests, small errors (.1 or .2 for $0 < \Theta \leq 1$) in predicting the optimum Θ-point of a particular power curve seemed of little consequence. It might be noted that Zelen and Joel [5], who work with a less precise estimate of power, present data which for various parameters reports Θ_p , the maximizing value of Θ , to lie in ranges of .2 to .8 .

Here the estimates of Θ_p found were considered accurate to .1 and these estimates were used to find an estimator which will predict Θ_p within .2 .

As was pointed out in Chapter I, generality is sacrificed to the extent of considering only cases in which $m_i/n_i = r$, $i = 1, 2$, and the data generated was concluded to be independent of r . Most of the data produced for this problem was with $r = 1$ or $r = 2$. That difference did not seem to cause any appreciable variation in Θ_p , nor did the few data sets produced with $r = 5$. It is quite possible that large values of r would noticeably effect Θ_p . However a large r means a very large number

of degrees of freedom and the power calculating technique used here was not sensitive enough to detect Θ_p accurately when the degrees of freedom of the two tests are large (except when λ_1 and λ_2 were very small), thus this check was not made.

In addition to the conclusion (1) that variations in r do not effect Θ_p , two further simplifying "conclusions" were reached. These are (2) for a particular pair of m_1 , m_2 values if $\lambda_1 = \lambda_2 = \lambda$, Θ_p varies little with λ , and (3) that the variation of Θ_p with m_1 and m_2 could be adequately described as a function of their ratio. Conclusion (2) can result in a .1 error as discussed in Chapter III. The magnitude of the error is a function of the spread of m_1 and m_2 (i.e., it is 0 when $m_1 = m_2$) but the nature of the functional relationship was not determined.

Conclusion (3), which considers m_2/m_1 sufficient for m_1 and m_2 can be evaluated by looking at Figure 2 which plots Θ_p against m_2/m_1 . It can be seen that no simple curve can fit these points. However, at this point in the data analysis, the errors due to precision of Θ_p and the first two simplifying conclusions are hopelessly confounded. The points of Figure 2 are approximated by a least squares fit and the factor $(m_2/m_1)^{.43}$ results.

The resulting form for Θ_p , $\hat{\Theta}_p = (m_2/m_1)^{.43} g(\lambda_1 , \lambda_2)$, (where $g(\lambda_1 , \lambda_2)$ is an unspecified function of λ_1 and λ_2) can best be evaluated by observing the correlation of the Θ_p(adj) values in Table II. By way of defense, it is said only that it seems to work.

The form of $g(\lambda_1 , \lambda_2)$ is another problem. For the application of Chapter IV, the use of the λ-ratio, $1-\lambda_2/1-\lambda_1$, is advocated because it is a reasonable approximation of $g(\lambda_1 , \lambda_2)$ over the range of λ_1 and λ_2 occurring from that application. For the general problem $f(\lambda_1 , \lambda_2)$ is presented in Table III. This $f(\lambda_1 , \lambda_2)$ is simply an averaging of various values produced from several computer runs incorporating the average of

Θ_p(adj) values from Table II.

It has been a temptation of say that $(1-\lambda_2)/(1-\lambda_1)$ describes all the systematic variation resulting from varying λ_1 and λ_2 . However, this is not the case. Use of the simply λ-ratio can introduce bias into the calculation of $\hat{\Theta}_p$ and will be magnified by a large spread of m_1 and m_2 .

Thus the conclusion must be a recommendation for the statistic $P_1 P_2^{(m_2/m_1)^{.43}(1-\lambda_2)/(1-\lambda_1)}$ accompanied by a word of caution on the use of $(1-\lambda_2)/(1-\lambda_1)$.

In Chapter V it is shown that this λ-ratio form of $\hat{\Theta}_p$ can be extended to combine three or more test statistics. The combined statistic which results has the property that all pairs of test are weighted optimally. Of course, the same word of caution must be injected because $(1-\lambda_2)/(1-\lambda_1)$ may introduce bias.

In many cases the abbreviated combined statistic $\prod_{i=1}^{n} P_i^{m_i^{.43}}$ can be used. This abbreviation is equivalent to saying $\lambda_i = \lambda_j$, $i \neq j$ and could be used in absence of any information on the magnitude of the λ_i , or where there is reason to believe that the various λ_i's are close to the same value (such as large numbers of blocks in the example of Chapter IV).

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