A Two-Factor Garma Long-Memory Model And Its Application To The Atmospheric CO_2 Data

by

Q.C. Cheng*, Wayne A. Woodward**, and H.L. Gray**

* University of Texas Southwestern Medical Center, Dallas, Texas
** Southern Methodist University, Dallas, Texas

Technical Report No. SMU/DS/TR-271

A TWO-FACTOR GARMA LONG-MEMORY MODEL AND ITS APPLICATION TO THE ATMOSPHERIC CO_2 DATA

Q.C. Cheng*, Wayne A. Woodward**, and H.L. Gray**

* University of Texas Southwestern Medical Center, Dallas, Texas

** Southern Methodist University, Dallas, Texas

ABSTRACT

Long-memory models have been used by several authors to model data with persistent autocorrelations. The fractional and FARMA models describe long-memory behavior associated with an infinite peak in the spectrum at $f = 0$. The Gegenbauer and GARMA processes of Gray, Zhang, and Woodward (1989) can model long-term periodic behavior for any frequency $0 \le f \le .5$. In this paper we introduce a two-factor extension of the Gegenbauer and GARMA models that allows for long-memory behavior to be associated with each of two frequencies in [0,.5]. We provide stationarity conditions for the two-factor model and discuss issues such as parameter estimation, model identification, realization generation, and forecasting. The use of the two parameter GARMA model is applied to the Mauna Loa atmospheric CO_2 data. It is shown that this model provides a reasonable fit to the CO_2 data, and it produces long-term forecasts that outperform those obtained via a fitted ARIMA model.

A TWO-FACTOR GARMA MODEL AND ITS APPLICATION TO THE ATMOSPHERIC CO_2 DATA[1]

Q.C. Cheng*, Wayne A. Woodward**, and H.L. Gray**

* University of Texas Southwestern Medical Center, Dallas, Texas
** Southern Methodist University, Dallas, Texas

1. Introduction

For many climate-related data sets, the correlation structure between observations may persist over long intervals of time. Such long-memory behavior was first noted by Hurst (1951) associated with the problem of determining reservoir storage capacity required to meet anticipated shortages. More recently, long-memory processes have been studied by a number of authors. A long-memory process has been defined as one for which $\sum_{i=1}^{\infty} |\rho_i|$ diverges, where ρ_i is the autocorrelation at lag i (McLeod and Hipel, 1978). Gray, Zhang and Woodward (1989) slightly extended this definition by defining a long-memory process as a process which has an unbounded power spectrum for some $f \in [0, .5]$.

In Section 2 we review long-memory models which have appeared in the literature. Section 3 describes the GARMA II model which is a new long-memory model capable of modeling the situation in which two persistent periodicities are present in the data. In Section 4 we apply the GARMA II model to the Mauna Loa CO_2 data

[1] This research was partially supported by DOE Environmental Sciences Division Grant DE-FG03-93ER61645.

(Keeling, et al., 1989) and compare its forecasts with those obtained by more traditional techniques.

2. Models for Long Memory Data

Models currently in use for describing time series with long-memory behavior are extensions of the ARMA(p,q) models popularized by Box and Jenkins (1976). The classical ARMA model is given by

$$\phi(B)(Y_t - \mu) = \theta(B)a_t \tag{1}$$

where

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$$

$$\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q ,$$

p, d, and q are nonnegative integers, a_t is zero mean white noise with variance σ_a^2, and B^k is the backward shift operator defined by $B^k f(t) = f(t-k)$. In general the pth order polynomial operator $\phi(B)$ can be factored as a product of first order and irreducible second order factors. The roots associated with a first order factor are real roots, and those associated with an irreducible second order factor appear as complex conjugate pairs. These first and second order factors serve as the "building blocks" determining the periodic behavior of the data. Each factor is associated with a "system frequency" . The system frequency, in cycles per year, associated with the first order factor $1-\alpha B$ is either 0 or .5 depending upon whether α is positive or negative respectively. The system frequency, f, associated with an irreducible second order factor $1-\alpha_1 B - \alpha_2 B^2$ is

$$f = \frac{1}{2\pi}\cos^{-1}\left(\frac{\alpha_1}{2\sqrt{-\alpha_2}}\right).$$

For a first order factor, the absolute value of the reciprocal of the associated root is $|\alpha_1|$ while for a second order factor it is $\sqrt{-\alpha_2}$. Factors associated with roots on or sufficiently near the unit circle (i.e. with absolute reciprocal near 1) dominate the behavior of the process. Woodward and Gray (1983, 1993) and Gray and Woodward (1986) suggest presenting this information in the form of a factor table.

An ARMA process is stationary if and only if the roots of characteristic polynomial $\phi(r) = 0$ all lie outside the unit circle. The most interesting class of nonstationary ARMA processes are those for which some roots of $\phi(r) = 0$ are on the unit circle (and none are inside the unit circle). When $\phi(r) = 0$ has a root of one, then $1 - B$ is a factor of $\phi(B)$. When there are d such unit roots, Box and Jenkins (1976) suggested expressing the model in the form

$$\phi_1(B)(1 - B)^d(Y_t - \mu) = \theta(B)a_t$$

where $\phi_1(B)$ is of order $p - d$. The notation typically used for an ARIMA(p,d,q) model is

$$\phi(B)(1 - B)^d(Y_t - \mu) = \theta(B)a_t \tag{2}$$

where $\phi(B)$ is of order p. Roots on the unit circle tend to dominate the behavior of a process, and <u>sample</u> autocorrelations from an ARIMA process with $d > 0$ will show very slow damping although the theoretical autocorrelations are constant. Gray and Woodward (1981) and Huang and Anh (1993) consider the more general ARUMA model which allows for complex as well as real roots on the unit circle. The general form of the ARUMA model is

$$\phi(B)\lambda(B)(Y_t - \mu) = \theta(B)a_t \tag{3}$$

where $\phi(B)$ has all of its associated roots outside the unit circle, $\lambda(B) = 1 - \lambda_1 B - \cdots - \lambda_s B^s$, and all of the roots of $\lambda(r) = 0$ lie on the unit circle. Thus, the operator $\lambda(B)$ can be expressed as a product of first order and second order factors associated with roots on the unit circle. Thus, it is clear that the ARUMA model contains the ARIMA models as a special case. A complex root on the unit circle will be associated with a second order factor $1 - \alpha_1 B - \alpha_2 B^2$ with $\alpha_2 = -1$. Such processes will be characterized by slowly damping sample autocorrelations that display a periodic behavior.

It should be noted that the nonstationary ARIMA and ARUMA processes are long memory. Hosking (1981, 1984) defined an extension of the ARIMA model which allows for the possibility of stationary, long-memory models. Specifically, these models are of the form

$$\phi(B)(1 - B)^{d}(Y_t - \mu) = \theta(B)a_t \tag{4}$$

where in this case d can take on fractional values. This process is called a fractional ARMA (FARMA) process. The process Y_t in (4) is stationary and long memory whenever $0 < d < 1/2$ and all of the roots of $\phi(r) = 0$ lie outside the unit circle. Spectral densities associated with the FARMA model are unbounded at $f = 0$. Gray, Zhang, and Woodward (1989) and Andel(1986) introduced an extension of the FARMA model which allows for long term dependency in stationary models associated with any frequency $f \in [0, .5]$. This extension is given by

$$\phi(B)(1 - 2uB + B^2)^{\lambda}(Y_t - \mu) = \theta(B)a_t \tag{5}$$

where u specifies the frequency at which the long memory behavior occurs, and λ essentially indicates how slowly the autocorrelations damp. The model is called the Gegenbauer ARMA (GARMA) due to the fact that $(1 - 2uB + B^2)^{-\lambda}$ is the generating function of the Gegenbauer polynomial, i. e.

$$(1 - 2uB + B^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(u)B^n \quad ,$$

where

$$C_n^{(\lambda)}(u) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(2u)^{n-2k}\Gamma(\lambda - k + n)}{k!(n-2k)!\Gamma(\lambda)}$$

is the Gegenbauer function (see e. g. Magnus *et al.*, 1966). The spectrum associated with (5) is

$$P(f) = \sigma_a^2 \frac{|\theta(e^{2\pi i f})|}{|\phi(e^{2\pi i f})|}\{4(\cos 2\pi f - u)\}^{-\lambda} \quad . \tag{6}$$

The frequency $f_0 = (\cos^{-1}u)/2\pi$ at which the spectrum becomes unbounded because of the Gegenbauer component is called the Gegenbauer frequency or G frequency. When $u = 1$ the GARMA model reduces to a FARMA model. For $|u| < 1$, Gray, Zhang, and

Woodward (1989) showed that the GARMA model is stationary and long memory whenever $0 < \lambda < 1/2$ and all of the roots of $\phi(r) = 0$ lie outside the unit circle. For $|u| < 1$ and $0 < \lambda < 1/2$, the autocorrelation function satisfies

$$\rho_k \sim k^{2\lambda-1}\cos(2\pi k f_0)$$

as $k \to \infty$ where f_0 is the G frequency. The notation $g_k \sim h_k$ as $k \to \infty$ indicates that $\lim_{k\to\infty}\{g_k/h_k\} = c$ where c is a finite, nonzero constant. Thus, the GARMA model is particularly appropriate for data with slowly damping autocorrelations which also have a cyclic pattern.

3. The GARMA II Model

The FARMA model can be thought of as an extension of the ARIMA model in that it models long memory behavior associated with a peak in the spectrum at $f = 0$. Similarly, the GARMA model relates to the ARUMA model in that it allows for long memory behavior associated with any frequency. However, a limitation of the FARMA model in (4) and the more general GARMA model in (5) is that they allow for only one stationary, long-memory component. Gray, Zhang, and Woodward (1989) suggested the inclusion of more than one Gegenbauer factor in the GARMA model, and Cheng (1993) has studied the following simplest of models in this extension:

$$\phi(B)(1 - 2u_1 B + B^2)^{\lambda_1}(1 - 2u_2 B + B^2)^{\lambda_2}(Y_t - \mu) = \theta(B) a_t \, . \tag{7}$$

We call this process a GARMA II process. The spectrum of the GARMA II process is given by

$$P(f) = \sigma_a^2 \frac{|\theta(e^{2\pi i f})|}{|\phi(e^{2\pi i f})|}\{4(\cos 2\pi f - u_1)^2\}^{-\lambda_1}\{4(\cos 2\pi f - u_2)^2\}^{-\lambda_2} \, . \tag{8}$$

From (8) it is clear that if $\lambda_i > 0$, then there is an unbounded peak in the spectrum at the frequencies $f_i = (\cos^{-1} u_i)/2\pi$, $i = 1,2$. As in the GARMA case, these two frequencies are called Gegenbauer frequencies, or simply G frequencies.

Note that the spectrum of the GARMA II involves the multiplication of the two Gegenbauer components in the model. However, even when the roots of $\phi(r)=0$ are all outside the unit circle, the multiplication of two stationary Gegenbauer factors in (7) does not guarantee the stationarity of the corresponding GARMA II process. For example, both of the factors $(1-1.6B+B^2)^{0.3}$ and $(1-1.6B+B^2)^{0.4}$ are associated with stationary Gegenbauer factors individually; however, their multiplication $(1-1.6B+B^2)^{0.3}(1-1.6B+B^2)^{0.4}$, which is a special case of the GARMA II model with $u_1=u_2$, does not create a stationary GARMA II process since the result of the multiplication is $(1-1.6B+B^2)^{0.7}$, a nonstationary Gegenbauer factor. The stationary region of the GARMA II processes is specified in the following theorem whose proof is given in Cheng, Woodward, and Gray (1994).

Theorem 1

A GARMA II process with all of the roots of $\phi(r)=0$ outside the unit circle, is stationary if one of the following conditions is satisfied.

(i) $|u_1|<1$, $|u_2|<1$, $u_1 \neq u_2$ and $\lambda_1<\frac{1}{2}$, $\lambda_2<\frac{1}{2}$

(ii) $u_1=u_2=u$, $|u|<1$ and $\lambda_1+\lambda_2<\frac{1}{2}$

(iii) $u_1=u_2=\pm 1$, and $\lambda_1+\lambda_2<\frac{1}{4}$

(iv) $u_1=1$, $u_2=-1$, $\lambda_1<\frac{1}{4}$ and $\lambda_2<\frac{1}{4}$

(v) $u_1=\pm 1$, $|u_2|<1$ and $\lambda_1<\frac{1}{4}$, $\lambda_2<\frac{1}{2}$

Cheng, Woodward, and Gray (1994) propose the following method for parameter estimation and model identification of a GARMA II model.

Step 1. Determine values of $\{\lambda_1, u_1, \lambda_2, u_2\}$ to be considered in a grid search by positions and magnitudes of the peaks in the spectrum.

Step 2. For each combination $\{\lambda_1, u_1, \lambda_2, u_2\}$, backward forecast the time points $-1, -2, \ldots, -M+1$ using a high order AR model, where M is a sufficiently large integer.

Step 3. Carry out the transformation

$$W_t=\left(\sum_{k=0}^{t+M-1} C_k^{(-\lambda_1)}(u_1)B^k . \sum_{l=0}^{t+M-1} C_l^{(-\lambda_2)}(u_2)B^l\right)Y_t$$

to obtain the approximately ARIMA process, W_t.

Step 4. Obtain the ARMA-based likelihood value for $\{W_t\}$.

Step 5. The combination $\{\lambda_1, u_1, \lambda_2, u_2\}$ which is associated with the largest likelihood value is the approximate maximum likelihood estimator.

Step 6. Calculate AIC (Akaike, 1974) for W_t based on the obtained approximate maximum likelihood estimates.

Step 7. To identify p and q, i. e. the order of $\phi(B)$ and $\theta(B)$, Step 1 through Step 5 can be repeated for different values of $\{p, q\}$. The final model is the one associated with the minimum AIC value.

Cheng, Woodward, and Gray (1994) demonstrated that the above procedure is effective for identifying the appropriate model and for estimating the parameters of a simulated GARMA II realization.

Forecasting future values is one of the major purposes of ARIMA and ARUMA models. For ARIMA processes the difference equation can be used to obtain the minimum mean square error forecast (Box and Jenkins, 1976). However, this method is not applicable to GARMA II processes because the order of the equation in this case is infinite. Cheng, Woodward, and Gray (1994) use the π-weights, i.e. the weights in the infinite order autoregressive expansion for Y_t in (7), i.e.

$$\theta^{-1}(B)\phi(B)(1-2u_1B+B^2)^{\lambda_1}(1-2u_2B+B^2)^{\lambda_2}\theta(Y_t-\mu)=a_t \,,$$

to obtain forecasts for a GARMA process, and in this case the forecast can be obtained as a weighted average of previous observations and forecasts made at previous lead times from the same origin t. The identity equation for the π weights is

$$\sum_{j=0}^{\infty}\pi_jB^j=\theta^{-1}(B)\phi(B)(1-2u_1B+B^2)^{\lambda_1}(1-2u_2B+B^2)^{\lambda_2}$$

$$=\theta^{-1}(B)\phi(B)\sum_{k=0}^{\infty}C_k^{(-\lambda_1)}(u_1)\sum_{l=0}^{\infty}C_l^{(-\lambda_2)}(u_2)\,.$$

The resulting forecasts can be written as

$$\hat{X}_{t_0}(h)=-\sum_{j=1}^{t_0-1+h}\pi_jX_{t_0+h-j}\,.$$

where the notation $\hat{X}_{t_0}(h)$ indicates a forecast if $h>0$ and the corresponding observed value if $h\le 0$. This method will be used in the next section to forecast the global carbon dioxide data.

4. Application to Atmospheric Carbon Dioxide Data

In this section we use the techniques presented here to model the carbon dioxide data. There are several sets of atmospheric carbon dioxide data available. The one we use here contains the largest such data set. It includes monthly measurements collected since March 1958 at the summit of Mauna Loa in Hawaii (Keeling, et al., 1989). Figure 1 shows the realization, autocorrelation and spectral density of the atmospheric carbon dioxide data. The slow decay rate of the autocorrelation function suggests that this is possibly a long memory case. In this section we apply the GARMA II process to atmospheric carbon dioxide data which has a multi-peak feature in its spectrum as shown in Figure 1.

To examine the basic feature of this data set, the method of overfitting an AR model discussed in Gray and Woodward (1986) is applied. Table 1 lists the factors of AR(20) and AR(25) fits to this data set using the GW time series computing package. In both cases the roots associated with the first three frequencies, namely 0.084, 0.0167 and 0.001, are very close to the unit circle. The first two correspond to the twelve and six months cycles since $1/0.084 \simeq 12$ and $1/0.167 \simeq 6$, while the third relates to the trend in the data, i.e. two unit roots. We will consider models for the CO_2 data which contains either the Gegenbauer factors $(1-1.732B+B^2)^{\lambda_1}$ and $(1-B+B^2)^{\lambda_2}$ or the nonstationary ARMA factors $(1-1.732B+B^2)$ and $(1-B+B^2)$. The third factor in the factor table will be included in our models via the factor $(1-B)^2$. Therefore the GARMA II model we suggest for this data set is

$$(1-2u_1B+B^2)^{\lambda_1}(1-2u_2B+B^2)^{\lambda_2}(1-B)^2\phi(B)(X_t-\bar{X}) = \theta(B)a_t \ ,$$

where $u_1 \simeq 0.866$, $u_2 \simeq 0.5$, and $\phi(B)$ and $\theta(B)$ are of orders p and q respectively.

The ML procedure described in the previous section for estimating λ_1 and λ_2 would require an excessive amount of computing time since we do not have prior knowledge of the model parameters p and q. For example the parameter space $\{0.2 \le \lambda_1 \le 0.49, \quad 0.2 \le \lambda_2 \le 0.49, \quad p \le 20, \quad q \le 20\}$ contains about one million combinations for calculating the MLE if the increments of λ_1 and λ_2 is 0.01. To reduce the computing time one may consider some further approximation of the estimation such as maximizing the likelihood based on estimates of λ_1 and λ_2 for a large value of p and taking $q = 0$, i. e. reducing the dimension of the parameter space. We did this for the subspace $\{0.2 \leq \lambda_1 \leq 0.49,\ 0.2 \leq \lambda_2 \leq 0.49,\ p = 1,\ q = 0\}$ and $\{0.2 \leq \lambda_1 \leq 0.49,\ 0.2 \leq \lambda_2 \leq 0.49,\ p = 20,\ q = 0\}$ which gave the estimates $\{\hat{\lambda}_1 = 0.21,\ \hat{\lambda}_2 = 0.27\}$ and $\{\hat{\lambda}_1 = 0.30,\ \hat{\lambda}_2 = 0.49\}$, respectively. To examine these estimates we have listed the factors of the data transformed by $(1-B)^2(1-1.7321B+B^2)^{0.30}(1-B+B^2)^{0.49}$ in Table 2. It can be seen that the factor associated with period of 12 months is still very strong after the transformation which would lead to a long-memory model which is dominated by the short memory behavior, i.e. by the factor $1-1.723B+.991B^2$ associated with the frequency $f = .083$. We desire to obtain a long-memory model which would explain the variation due to the dominant frequencies within the long-memory components. In order to accomplish this we estimate λ_1 and λ_2 to be those values of λ which minimize the presence of $f_1 = \cos^{-1}u_1/2\pi$ and $f_2 = \cos^{-1}u_2/2\pi$ in the periodogram of the residuals of the original data. Since the periodogram at a frequency represents the amount of variance associated with that frequency in the data, the rationale behind this criterion can be understood as reducing the effect of these two frequencies in the data set as much as possible so that the related near nonstationary behavior can be removed. Using $\hat{\lambda}_1 = .49$, $\hat{\lambda}_2 = .49$, $u_1 = .866$, $u_2 = .5$, and $(1-B)^2$ the data are transformed to an ARIMA process.

The data transformed by the Gegenbauer II component and the factor $(1-B)^2$ are then modeled as an AR(17) based on standard ARMA model identification techniques such as AIC (Akaike, 1974) and array methods such as GPAC (Woodward and Gray, 1981). Thus the final GARMA II model for this data set is

$$(1-1.732B+B^2)^{0.49}(1-B+B^2)^{0.49}(1-B)^2\hat{\phi}(B)(X_t-\hat{\mu}_x) = a_t$$

where $\hat{\phi}(B)$ represents the maximum entropy estimates (Burg, 1975)

$$\begin{aligned}\hat{\phi}(B) = 1 &+ 2.32B + 3.23B^2 + 3.52B^3 + 3.42B^4 + 3.27B^5 + 3.25B^6 \\ &+ 3.31B^7 + 3.29B^8 + 3.09B^9 + 2.86B^{10} + 2.54B^{11} + 1.83B^{12} \\ &+ 0.91B^{13} + 0.17B^{14} - 0.21B^{15} - 0.20B^{16} - 0.08B^{17}\end{aligned}$$

with $\hat{\mu}_x = 327.57$ and $\hat{\sigma}^2{}_a = 1.5$.

Forecasts for this GARMA II model are shown in Figure 2 where the forecasts and the actual data are represented by the solid line and the dashed line, respectively. The FORTRAN program used for this purpose is available in Cheng (1993). It can be seen that the forecasts are excellent, having the same cycles and the upward trend present in the data.

As an alternative model for the carbon dioxide data, we change the values of λ_1 and λ_2 to one, which changes the GARMA II process to a nonstationary ARMA process

$$(1-1.732B+B^2)(1-B+B^2)(1-B)^2\phi_1(B)(X_t-\bar{X})=\theta_1(B)a_t$$

Applying the GPAC and S-array (Woodward and Gray, 1981) to the data transformed by the two nonstationary second order factors and $(1-B)^2$, we identify the transformed data as a stationary AR(11). The Burg procedure gives the following estimates of the coefficients:

$$\begin{aligned}\hat{\phi}(B) = \ & 1+3.64B+6.99B^2+8.92B^3+7.84B^4+4.07B^5-0.28B^6\\ & -2.99B^7-3.29B^8-2.14B^9-0.90B^{10}-0.20B^{11}\end{aligned}$$

Figure 3 shows the forecasts for the same 60 observations considered in the GARMA II forecasts. The forecasts have the same cycle as the data, but they fail to follow the same upward trend after one or two cycles.

Table 3 lists the mean square error of the forecasts from six different forecast origins for both the GARMA II and ARMA models. The ARMA model has smaller MSE in short term forecasts while the GARMA II model performs better in the long term.

From the analysis carried out above one can see that the long memory GARMA II is a reasonable model for the global carbon dioxide data. One can also conclude that for the long term forecasts considered here the GARMA II model is more suitable than the ARMA model with nonstationary components. Also, it should be noted that based on either model, if conditions remain unchanged, the upward trend is predicted to continue.

REFERENCES

Akaike, H. (1974). "A New Look at the Statistical Model Identification," *IEEE Transactions on Automatic Control* 19, 716-723.

Andel, J. (1986). "Long Memory Time Series Models," *Kybernetika* **22**, 105-123.

Box, G. E. P. and Jenkins, G. M. (1970). *Time Series Analysis: Forecasting and Control,* San Francisco: Holden-Day.

Burg, J.P.. "Maximum Entropy Spectral Analysis," Ph.D. Dissertation, Stanford University.

Cheng, Q. C. (1993). "Transfer Function Model and GARMA II Model," Ph.D. Dissertation, Southern Methodist University.

Cheng, Q.C., Woodward, W.A., and Gray, H.L. (1994). "A Two-Factor GARMA Long-Memory Model," submitted to *Journal of Time Series Analysis.*

Gray, H, L. and W. A. Woodward (1986). "A New ARMA Spectral estimator," *Journal of the American Statistical Association* 81, 1100-1108.

Gray, H. L., Nien-Fan Zhang and W. A. Woodward (1989). "On Generalized Fractional Processes," *Journal of Time Series Analysis* **10,** 233 – 257.

Hosking, J. R. M. (1981a). "Fractional Differencing," *Biometrika* **68,** 165 – 176.

Hosking, J. R. M. (1981b). "Some Models of Persistence in Time Series," *Time Series Analysis: Theory and Practice 1, Proceedings of the International Conference held at Valencia,* ed. O. D. Anderson, 641 – 653, Amsterdam: North – Holland.

Hosking, J. R. M. (1984). "Modeling Persistence in Hydrological Time Series Using Fractional Differencing," *Water Resources Research* **20**, 1898 – 1908.

Huang, D. and V. V. Anh (1993). "Estimation of the Non-stationary Factor in ARUMA Models," *Journal of Time series analysis* **14**, 27-46.

Hurst, H. E. (1951). "Long-term Storage Capacity in Reservoirs," *Proceedings of the Institute of Civil Engineering* **116**, 770-808.

Keeling, C. D., R. B. Bacastow, A. F. Carter, S. C. Piper and T. P. Whorf (1989). *American Geophysical Union, Geophysics Monograph* **55**, 165-236.

Mcleod, A. I. and K. W. Hipel (1978). "Preservation of the Rescaled Adjusted Range, 1. A Reassessment of the Hurst Phenomenon," *Water Resources Research* **14**, 491-508.

Magnus, W., F. Oberhettinger and R. P. Soni (1966). *Formulas and Theorems for the Special Functions of Mathematical Physics,* Berlin: Springer-Verlag.

Woodward, W.A. and Gray, H.L. (1981). "On the Relationship Between the S-array and the Box-Jenkins Approach to ARMA Model Fitting," *Journal of the American Statistical Association, 76*, 579-587.

Woodward, W. A. and H. L. Gray (1983). "A Comparison of Autoregressive and Harmonic Component Models for the Lynx Data, *Journal of the Royal Statistical Society A* **146**, 71-73.

Woodward, W. A. and H. L. Gray (1993). "Global Warming and the Problem of Testing for Trend in Time Series Data," *Journal of Climate* **6**, 953-962.

TABLE 1

FACTOR TABLE FOR GLOBAL CARBON DIOXIDE DATA

$p = 20$

Absolute reciprocal of root	Frequency	Factors
0.999	0.084	$1 - 1.729B + 0.999B^2$
0.997	0.167	$1 - 0.997B + 0.993B^2$
0.995	0.001	$1 - 1.990B + 0.990B^2$
0.931	0.243	$1 - 0.079B + 0.867B^2$
0.916	0.339	$1 + 0.973B + 0.839B^2$
0.889	0.414	$1 + 1.526B + 0.790B^2$
0.863	0.472	$1 + 1.698B + 0.744B^2$
0.750	0.278	$1 + 0.264B + 0.563B^2$
0.748	0.110	$1 - 1.151B + 0.560B^2$
0.487	0.369	$1 + 0.660B + 0.237B^2$
$p = 25$		
1.000	0.084	$1 - 1.729B + 0.999B^2$
0.998	0.167	$1 - 0.998B + 0.997B^2$
0.994	0.001	$1 - 1.988B + 0.989B^2$
0.964	0.337	$1 + 0.996B + 0.928B^2$
0.957	0.249	$1 - 0.013B + 0.917B^2$
0.951	0.421	$1 + 1.671B + 0.905B^2$
0.927	0.378	$1 + 1.334B + 0.860B^2$
0.926	0.465	$1 + 1.808B + 0.858B^2$
0.925	0.126	$1 - 1.300B + 0.856B^2$
0.920	0.292	$1 + 0.484B + 0.846B^2$
0.913	0.217	$1 - 0.379B + 0.834B^2$
0.885	0.500	$1 + 0.885B$
0.821	0.042	$1 - 1.586B + 0.674B^2$

TABLE 2

FACTOR TABLE FOR GLOBAL CARBON DIOXIDE DATA

TRANSFORMED BY $(1-B)^2(1-1.7321B+B^2)^{0.30}(1-B+B^2)^{0.49}$

$p = 20$

Absolute reciprocal of root	Frequency	Factors
0.995	0.083	$1-1.723B+0.991B^2$
0.953	0.241	$1-0.105B+0.909B^2$
0.940	0.334	$1+0.948B+0.884B^2$
0.930	0.167	$1-0.922B+0.864B^2$
0.920	0.421	$1+1.619B+0.846B^2$
0.911	0.473	$1+1.797B+0.830B^2$
0.889	0.278	$1+0.313B+0.790B^2$
0.865	0.375	$1+1.226B+0.748B^2$
0.747	0.131	$1-1.014B+0.558B^2$
0.624	0.000	$1-0.623B$
0.538	0.500	$1+0.538$

TABLE 3

MEAN SQUARE ERROR FOR THE GARMA II AND ARMA MODELS FOR THE Mauna Loa CARBON DIOXIDE DATA

	Forecast Origin t_0					
Model	370	360	350	340	330	320
GARMA II	1.48	1.28	5.85	2.05	2.70	0.47
ARMA	0.62	0.17	1.98	3.73	6.49	5.83

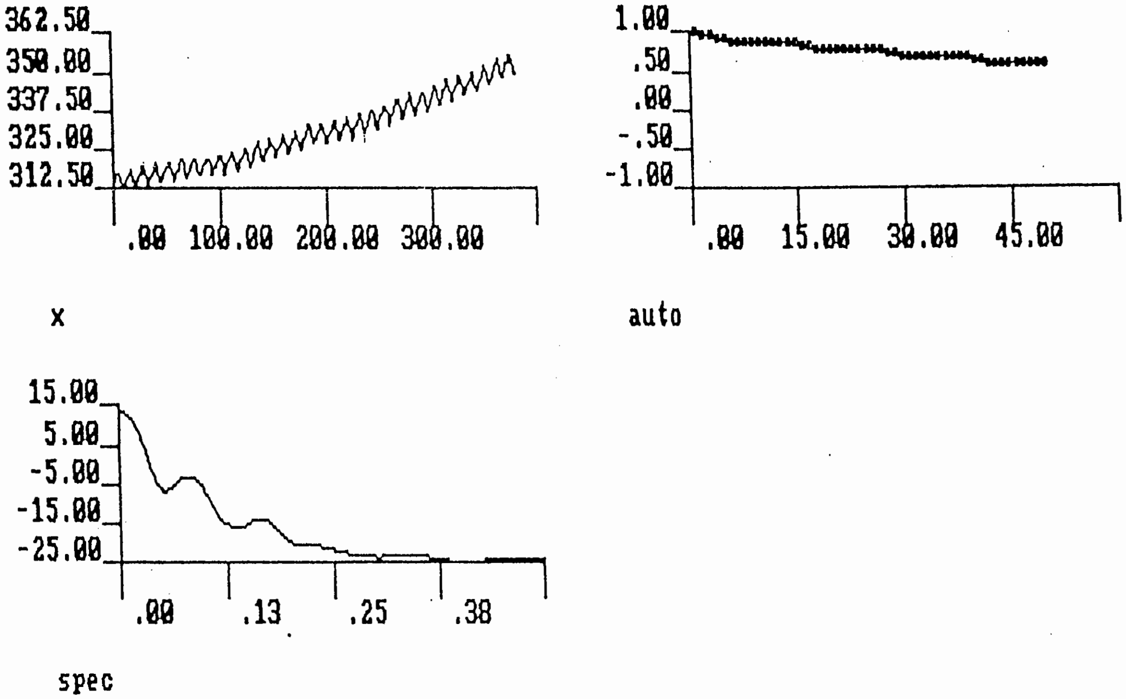

FIGURE 1. The realization, autocorrelation and spectral density of the Global carbon dioxide data

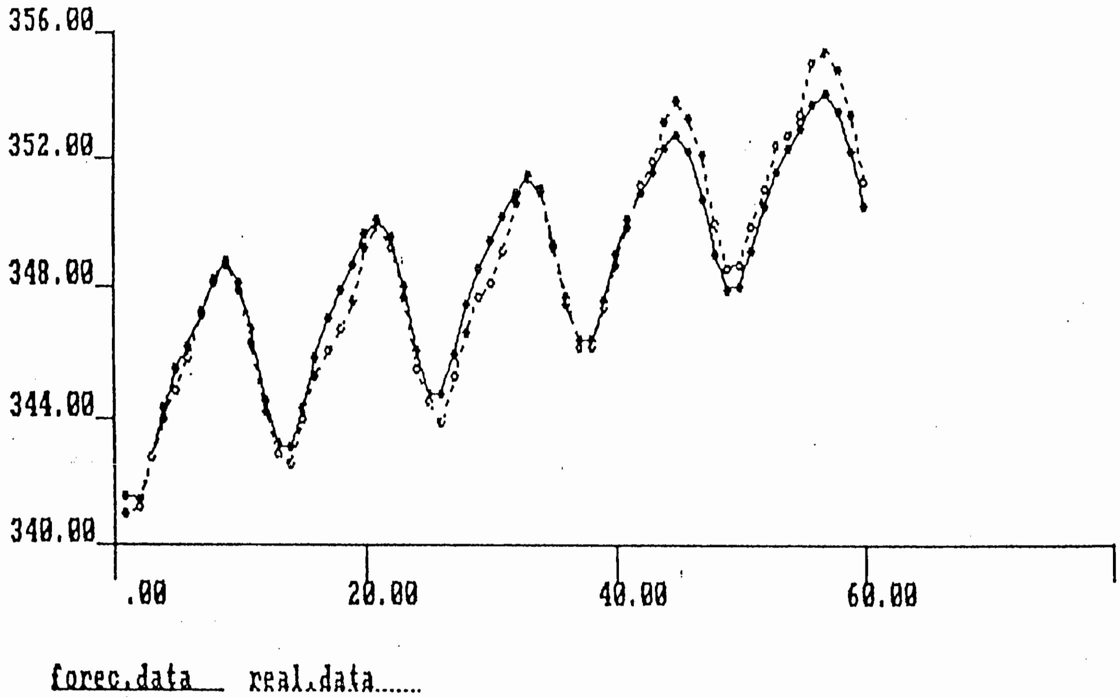

FIGURE 2. Forecasts of the last 60 observations of global carbon dioxide data based on a GARMA II model

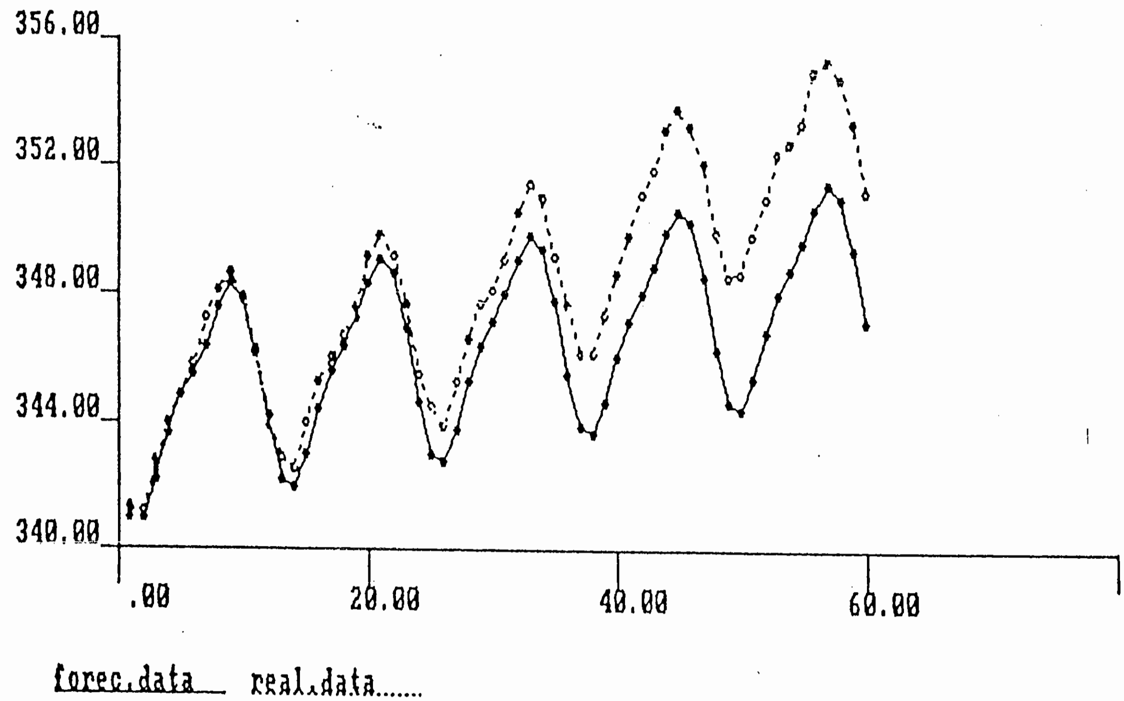

FIGURE 3. Forecasts of the last 60 observations of global carbon dioxide data based on an ARMA model