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DISCRETE TWO-PERSON GAME THEORY
WITH MEDIAN PAYOFF CRITERION

by

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DISCRETE TWO-PERSON GAME THEORY WITH MEDIAN PAYOFF CRITERION

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ABSTRACT

The minimax concept for solution of discrete two-person games is based on expected value considerations and has a zero-sum condition for payoffs. This approach often is inapplicable due to strong violation of the zero-sum condition. By use of a different criterion, based on median value considerations, a two-person game theory can be developed that seems appropriate for a class of situations called competitive. It is also applicable for a hugely broader class of practically important situations called median competitive. Consider cases where each player is either protective toward himself or vindictive toward the other player. A largest value P_I (P_{II}) occurs in the payoff matrix for protective player I (II) such that he can assure himself at least this payoff with probability at least 50 percent. A smallest value P'_I (P'_{II}) occurs in the matrix for player I (II) such that vindictive player II (I) can assure, with probability at least 50 percent, that player I (II) receives at most this payoff. For competitive and median competitive games, a player is simultaneously protective and vindictive. Values of P_I, P_{II}, P'_I, P'_{II}, and median optimum strategies are nearly always determined without great effort. This can be done by solution of zero-sum games (expected value basis) with identified payoff matrices containing only ones and zeroes. Deciding on payoff values is simplified for the median approach. Except for P_I, P_{II}, P'_I, and P'_{II} it is sufficient to know the relative order of the values for each payoff matrix. The median approach has the strong practical advantage of being applicable even when payoffs in different matrices cannot meaningfully be added or subtracted (such as when only relative ordering is known for one or both matrices).

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INTRODUCTION AND DISCUSSION

This paper presents a form of discrete two-person game theory that is based on median value considerations (motivated by the median estimation concept in statistics). Two extreme situations arise, depending on whether a player is acting protectively for himself or vindictively towards the other player. A protective player is interested in the largest payoff such that he can assure himself at least this value with a probability of at least 50 percent. A vindictive player is interested in the smallest payoff such that he can assure, with a probability of at least 50 percent, that the other player receives at most this value. A class of "Median Optimum" strategies can be defined, for either the protective or vindictive approach.

For a class of games called "competitive" (also for a much broader class called "median competitive"), there exist strategies which are simultaneously median optimum from the protective viewpoint and from the vindictive viewpoint. Consequently, for those games, identification of a median optimum strategy as protective or vindictive is unnecessary. In spite of certain parallels between the median-optimum viewpoint and the minimax viewpoint, a pure strategy may be median optimum without necessarily being a minimax strategy.

The first introductory material outlines the median approach and some of its properties. Then, some comparisons are made with the standard minimax form of two-person game theory.

Each player has a separate matrix that states the payoff he receives for each combination of a pure strategy of his with a pure strategy of the other player. Both of these matrices are considered to be known to each player. The pair of payoffs to the players that occurs for a given combination of a pure strategy for each player is an outcome of the game. The first value in an outcome is the payoff to player I and the second value is the payoff to player II.

Each player then imposes a preference-ordering on the outcomes. The preference-ordering of a protective player I is considered to be such that his own payoffs are nondecreasing and, by convention, the preference-ordering of the protective player II is such that his own payoffs are nonincreasing. In contrast, a vindictive player I imposes an ordering such that the payoffs to player II are nonincreasing, and a vindictive player II imposes an ordering where the payoffs to player I are nondecreasing. If there are tied values among the payoffs considered, there may be many alternative orderings satisfying any one of those conditions. A vindictive player could order within ties so as to be as advantageous to himself as possible. A protective player could order within ties so as to be as disadvantageous to the other player as possible. On the other hand, a protective player could order within ties so as to be as advantageous to the other player as possible, etc. Finally, more than one outcome could possibly have the same pair of payoffs. A player may select among the possible orderings of such "double ties" on the basis of the strategy combinations corresponding to these outcomes. In all cases, each player (identified as protective or vindictive) chooses a sequence that is called his preferred sequence. The preferred sequences provide the basis for application of the median approach.

A game is said to be competitive if the outcomes can be sequence ordered so that the payoffs for player I are nondecreasing and the payoffs for player II are nonincreasing. The possible ordering of outcomes is unique when the payoffs of player I are strictly increasing or those of player II are strictly decreasing; then we assume that the preference orderings of the two players are the same. However, more than one eligible sequence order is possible when ties in payoff value occur for both players. That is, the same outcome value could possibly occur for more than one combination of strategies. Among these sequences (which are the same in values of outcomes), each player selects his preferred sequence on the basis of the strategy combinations corresponding to the pertinent tied outcomes and from related probability considerations. If the preferred sequence is still not unique, it can be chosen arbitrarily.

A largest payoff P_I (P_{II}) occurs in the matrix for player I (II) such that he can assure at least this value with a probability of at least 50 percent. A smallest payoff P'_I (P'_{II}) occurs in the matrix for player I (II) such that player I (II) receives at most this amount.

Let us consider determination of P_I (P_{II}) for a protective player I (II). There exists a subset of outcomes in the preferred sequence for protective player I (II) that consists of a determined outcome and all outcomes above (below) it. Occurrence of an outcome in this identified subset can be assured with a probability of at least 50 percent, but such is not the case when the determined outcome is removed. A protective player is considered to select his preferred sequence so as to minimize the number of outcomes in the identified subset. When more than one eligible subset has the minimum number of outcomes, a subset that has the highest assured probability is used for the preferred sequence. The value of P_I (P_{II}) is the payoff for player I (II) in the lowest (highest) outcome of his identified subset. Incidentally, one or more outcomes consecutively adjacent to the identified subset could also have payoff P_I (P_{II}) for player I (II).

Now, consider determination of the payoff P'_{II} (P'_I), in the other player's matrix, that is associated with a vindictive player I (II). There exists a subset of outcomes in the preferred sequence for vindictive player I (II) that is identified in the same way as was given for evaluating P_I and P_{II}. The value of P'_{II} (P'_I) is the payoff to player II (I) in the lowest (highest) outcome of the identified subset for player II (I).

Detailed statement of results for the general case occurs in the next section. However, some additional information about results for competitive games is given here. When each player has a pure median optimum strategy, player I (II) can guarantee (100 percent probability) at least $P_I = P'_I$ ($P_{II} = P'_{II}$). When player I (II) has a pure median optimum strategy but player II (I) does not, player I (II) can guarantee at least P_I (P_{II}), and player II (I) can assure at least P'_{II} (P'_I) with a probability greater than 50 percent. When neither player has a pure median optimum strategy, player I (II) can assure that he receives at least P_I (P_{II}) with probability at least 50 percent, and that player II (I) receives at most P'_{II} (P'_I) with probability at least 50 percent.

It is instructive to consider some characteristics of the median of a probability distribution for a situation where both players use mixed strategies. The median value of a distribution is not necessarily unique. That is, all the permissible values in an extensive interval can be medians of a probability distribution. This property can be convenient. Suppose that the game is competitive and consider the set of median values for player I (II). Also, suppose that both players use mixed strategies that are optimum for the median approach. Then, all payoffs that are at least equal to P'_I (P'_{II}) and at most equal to P_I (P_{II}) are medians of the probability distribution of the payoff to player I (II). Thus, as would be expected in competitive situations, player I (II) seeks to maximize the upper payoff P_I (P_{II}) in his set of median values and to minimize the lower payoff P'_{II} (P'_I) in the set of median values for player II (I). The properties of a median allow this to be done simultaneously in such a way that P_I (P_{II}) and P'_I (P'_{II}) can be far apart in an ordering of the payoff values for player I (II). It is thus possible for P_I to be in the upper payoff values for player I simultaneously with P_{II} being in the upper values for player II. In fact, this seems to occur for many kinds of combinations of payoff matrices for players I and II in competitive games.

Required information about payoff matrices is not very great when the median approach is used. It is sufficient to first determine the relative order (includes equality) of the values for each matrix, which determines the locations of P_I, P'_I, P_{II}, and P'_{II}. Deciding on values for P_I, P'_I, P_{II}, and P'_{II} completes the information that is required about the payoff matrices.

It is more difficult to determine the appropriateness of the median approach when the game is not even roughly of a competitive nature. Then, a low payoff for one player does not necessarily correspond to a high payoff for the other. Thus, the median payoff, say to player I, might be substantially different for the protective and vindictive situations. Also, cases where cooperation would increase the payoff to both players can occur. However, the median approach can be useful when the players are not allowed to communicate (so that a player only knows his own payoff matrix). Also, results like those developed for competitive games can be obtained for a rather broad class of situations that are termed median competitive.

The most general form of median competitive games, and corresponding properties, have not been determined yet and provide a subject for future investigation. An example of games that are median competitive, but not necessarily competitive, is given here. The example consists of all games that "generate" competitive games. A game is said to generate a competitive game if, for both players, there exist sequences that are eligible to be preferred sequences and for which the following two conditions are satisfied: First, the payoffs of player I (II) that are in outcomes above (below) the outcome determining P_I (P_{II}) are at least (most) equal to P_I (P_{II}), and the payoffs in outcomes below this outcome are at most (least) equal to P_I (P_{II}). Second, the payoffs of player I (II) that are below (above) the outcome determining P'_I (P'_{II}) are at least (most) equal to P'_I (P'_{II}), and the payoffs in outcomes below this outcome are at most (least) equal to P'_I (P'_{II}). Then, new outcomes can be formed, by pairing the payoffs of player I with those of player II, that satisfy the requirements for a competitive game but leave the outcomes that determined P_I, P_{II}, P'_I, P'_{II} fixed and at the same sequence positions. This is done so that the groups of payoffs (without regard to order) in the identified subsets for the competitive game are the same as the groups in these subsets for the original game. Since the results developed depend only on the outcomes that determine P_I, P_{II}, P'_I, P'_{II} and on the groups of payoffs in the identified subsets, the results for this competitive game also apply to the game from which it was generated.

Perhaps the most attractive feature of the median approach is its ability to handle competitive and median competitive games where payoffs from different matrices cannot be meaningfully added or subtracted. This permits use of a game theory approach for an important and extensive class of situations. In fact, many of the situations occurring in economics and the social science areas (psychology, education, etc.) have matrices of this kind, maybe due to the fact that only relative order can be determined within a payoff matrix. However, situations of this class occur in virtually all areas where game theory is potentially useful (including military applications).

Now let us compare the median approach with the minimax procedure where a zero-sum condition is imposed on the two payoff matrices and the criterion is the expected value of the payoff to player I. Discussion of expected value and median value properties occurs first.

The outcome that results when one or both players use a mixed strategy is a random value. This outcome can be identified by a representative property of its probability distribution. The mean of this distribution (expected value of the random outcome) is one representative property that could be considered. The distribution median (not necessarily unique) is another representative property that is useful. Each of these properties has desirable and undesirable features. Neither has been shown to be uniformly preferable to the other. The median is especially appropriate when there is not much interest in the extensiveness of distribution tails. The mean can be preferred when large deviations from the central part of the distribution are important, even though their probability of occurrence is very small. Often, choice of whether to consider the expected value or the median value is based on the utility and convenience aspects of the situation. If, for the situation considered, useful and much more extensive results can be obtained for the median, statisticians seldom hesitate to consider it rather than the distribution mean.

Discrete two-person game theory is a case where median considerations seem to lead to more extensive results of a worthwhile nature than do expected value considerations. The median approach is applicable for the rather broad class of competitive and median competitive games, including games where one or both payoff matrices have ordinal numbers. The minimax approach required cardinal numbers in both matrices but still only applies to the small subclass of competitive games where the matrices at least roughly satisfy a zero-sum condition. Values must be determined for all (or nearly all) of the outcomes when the minimax approach is used. Except for a few payoffs (usually four, and never more than four) only relative order among the payoffs in each matrix must be determined for the median approach.

For games of a zero-sum type, it would seem that a combined use of the expected value criterion and the median approach could be desirable. That is, the strategy used by a player is at least approximately optimum in an expected value sense and also assures at least an identified payoff with a probability that has a lower bound not greatly below 50 percent. The resulting median payment would ordinarily be less than P_I for player I and less than P_{II} for player II. Such strategies are especially desirable when values of payoffs are only roughly known but relative ordering is precisely known within each matrix. The determination of strategies with these combined properties is another subject for future investigation.

Payoff values that represent "catastrophe" can appear in a payoff matrix. A modification of the median approach is needed to avoid the occurrence of such extreme payoffs. One possible method is to not use any row that contains a catastrophic payoff. The usual median approach would be applied to the payoff matrices resulting when these rows (columns in the matrix for the other player) are removed. However, further investigations to obtain suitable modifications would seem to be needed.

Only discrete games are considered here. However, extension of the median approach to continuous cases, and combinations of continuous and discrete cases, seems definitely possible and worthwhile. This extension is a further subject for future investigation.

The next section contains statements of how to determine P_I, P'_I, P_{II}, P'_{II}, and optimum strategies for each player. Also, properties of results using the median approach are stated more precisely. The final section contains the basis for the results (in terms of three theorems).

RESULTS

Let the payoff matrix for player I (II) be stated so that rows represent pure strategies for player I (II) and columns are pure strategies for player II (I). For all applications, a marking of some of the values in the payoff matrices is made initially, with this being done separately for each matrix. The case of a protective player is considered first.

For protective player I (II), first mark the position, in his matrix, of his payoff in the last (first) outcome of his preferred sequence of outcomes. Then do this for the next to the last (first) outcome for player I (II), etc. Continue consecutively in his (protective) preferred sequence of outcomes until the first time that this player can assure obtaining a marked value with probability at least 1/2. The value of P_I (P_{II}) is the last payoff marked in the matrix for player I (II). For competitive games, P'_I (P'_{II}) is the payoff to player I (II) in the last outcome that was marked in the matrix of player II (I).

Determination of P_I (P_{II}), and the corresponding pairs, is perhaps best accomplished by initially marking the matrix for player I (II) until the first time that two or fewer rows contain marks in all the columns. (The value of P_I (P_{II}) is greater than or equal to the last payoff marked in this manner, and can be greater; based on Theorem 1.) Next, working forward (backward) in the preferred sequence for protective player I (II), remove the mark from the payoff (unique) that was marked last. Then, replace the remaining marked values with ones and replace all other payoffs in the matrix by zeroes. Consider this matrix of ones and zeroes to be for a zero-sum game with an expected value basis. Solve for the value of the game. If this game-value is less than 1/2, the marking is completed by again marking the payoff whose mark was removed (which also determines the corresponding outcome). If the game-value is at least 1/2, continue in the same way (removing the mark from the last payoff that was considered among those still marked, forming a matrix with ones and zeroes, etc.). If the resulting game-value is less than 1/2 the payoff whose mark was last removed is marked again and the marking is completed. This marking procedure is continued until a game-value less than 1/2 occurs. (This procedure, and that in the next paragraph, are based on Theorem 2.) From examples, it seems that P_I and P_{II} are often the payoffs that resulted in the first time that two or fewer rows contain marked values in all columns of the respective matrices.

The zero-sum game (matrix of ones and zeroes) that occurs for the final marking in evaluating P_I or P_{II} is also used to determine (protective) median optimum strategies for the player with that matrix. That is, an optimum strategy of this player for that game is also a median optimum strategy. In particular, consider the situation for player I (II) when P_I (P_{II}) happens to be the payoff whose marking resulted in a pair of rows that contain marked values in all columns (but no fully marked row occurs). Examination of the zero-sum game shows that a mixed median optimum strategy for player I (II) consists in choosing one of the rows of this pair with probability 1/2 for each row.

For player I (II) vindictive, first mark the position in the matrix for player II (I) that is in the last (first) outcome of the preferred sequence of player I (II). Then do this for the next to last (first) outcome for player I (II). Continue consecutively in the (vindictive) preferred sequence for player I (II) until the first time that he can assure obtaining a marked value in the matrix of player II (I) with probability at least 1/2. The value of P'_{II} (P'_{I}) is the last payoff marked in the matrix for player II (I).

Determination of P'_{II} (P'_{I}) can be accomplished by initially marking the matrix for player II (I), according to the vindictive preferred sequence for player I (II), until the first time that two columns contain marks in all the rows. Next, remove the mark from the payoff that was marked last. Replace the remaining marked values with ones and all other payoffs by zeroes. Consider the resulting matrix to be for a zero-sum game and solve for the game-value. If this game-value is greater than 1/2, the marking is completed by again marking the payoff whose mark was removed. If the game-value is at most 1/2, continue in the same way with removal of another mark. If the resulting game-value is greater than 1/2, again mark the payoff whose mark was last removed and the marking is completed. This marking procedure is continued until a game-value greater than 1/2 occurs. As for the protective case, it seems that P'_{I} and P'_{II} are often the payoffs that resulted the first time that two or fewer columns contained marked values in all rows.

The zero-sum game that occurs for the final marking of the matrix for player II (I) can be used to determine (vindictive) median optimum strategies for player I (II). That is, an optimum strategy of player I (II) for this game, that is based on the matrix for player II (I), is also a median optimum strategy. When P'_{II} (P'_{I}) happens to be the payoff that resulted in a pair of columns with marks in all rows (but no fully marked column occurs), a mixed median optimum strategy for player I (II) consists in selecting one of these two columns with probability 1/2 for each column.

Statement of results occurs next. Cases where pure median optimum strategies occur are considered first. In all cases, a protective player I (II) can guarantee himself at least P_I (P_{II}) by using the fully marked row in his matrix. A vindictive player I (II) can always guarantee that player II (I) receives at most P'_{II} (P'_I) by using the fully marked column in the matrix for player II (I).

Now, consider the case where each player has a pure median optimum strategy. When a protective player I (II) uses the fully marked row in his matrix and a vindictive player II (I) uses the fully marked column in the matrix for player I (II), player I (II) receives exactly $P_I = P'_I$ ($P_{II} = P'_{II}$) and player II (I) receives the payoff in his matrix that corresponds to the strategy combination for this row and column. When protective players I and II both use fully marked rows, player I sometimes receives more than P_I and/or player II sometimes receives more than P_{II}. When vindictive players I and II both use fully marked columns (in the other player's matrix), player I sometimes receives less than P'_I and/or player II sometimes receives less than P'_{II}/ When the game is competitive and each player has a pure median optimum strategy, $P_I = P'_I$ and $P_{II} = P'_{II}$; also, when each player uses his pure median optimum strategy, player I (II) receives P_I (P_{II}).

Now, consider the case where a pure median optimum strategy occurs for player I (II) but not for player II (I). First, suppose that player I (II) is protective. Then, player I (II) can guarantee himself at least P_I (P_{II}), and a protective player II (I) can assure himself at least

P_{II} (P_I) with a probability of at least 1/2. Player I (II) can also guarantee himself at least P_I (P_{II}) against a vindictive player II (I), and player II (I) can assure that player I (II) receives at most $P_I = P'_I$ ($P_{II} = P'_{II}$) with a probability greater than 1/2 (Theorem 3). Next, suppose that player I (II) is vindictive. Then, player I (II) can guarantee that a protective player II (I) receives at most $P'_{II} = P_{II}$ ($P'_I = P_I$), and player II (I) can assure himself at least P_{II} (P_I) with a probability greater than 1/2 (Theorem 3). Player I (II) can also guarantee that vindictive player II (I) receives at most P'_{II} (P'_I), and player II (I) can assure, with probability at least 1/2, that player I (II) receives at most P'_I (P'_{II}). Now consider competitive games. Then, $P_I = P'_I$ and $P_{II} = P'_{II}$ (Theorem 3). Player I (II) can guarantee that he receives at least P_I (P_{II}) and that player II (I) receives at most P_{II} (P_I). Player II (I) can assure that he receives at least P_{II} (P_I) and also that player I (II) receives at most P_I (P_{II}), with a probability greater than 1/2.

Finally, consider the case where no pure median optimum strategy occurs for either player. Suppose that both players are protective. Then, player I (II) can assure at least P_I (P_{II}) with a probability of at least 1/2. When player I (II) is protective and player II (I) is vindictive, player I (II) can assure that he receives at least P_I (P_{II}) with probability at least 1/2 and player II (I) can assure that player I (II) receives at most P'_I (P'_{II}) with probability at least 1/2; these probabilities are exactly 1/2 when both players use mixed median optimum strategies. Next, suppose that both players are vindictive. Then, player I (II) can assure that player II (I) receives at most P'_{II} (P'_I) with a probability of at least 1/2. Now consider competitive games.

Player I (II) can simultaneously assure, with probability at least 1/2, that he receives at least P_{I} (P_{II}) and that player II (I) receives at most P'_{II} (P'_{I}). When both players use mixed median optimum strategies, player I (II) receives at least P_{I} (P_{II}) with probability exactly 1/2 and at most P'_{I} (P'_{II}) with probability exactly 1/2.

BASIS FOR RESULTS

The procedure for determining P'_I, P'_{II}, and vindictive median optimum strategies can, with suitable interpretation be obtained directly from that for determining P_I, P_{II}, and protective median optimum strategies. Hence, verification for the protective case is sufficient.

The results for both players protective, or both vindictive, can be directly verified from the properties of the procedures for determining P_I, P_{II}, P'_I, and P'_{II}. This is also the case for one player protective and the other vindictive when both players have pure median optimum strategies or neither player has a pure median optimum strategy.

A competitive game can be considered to be a combination of the situation where player I is protective and player II vindictive with the situation where player I is vindictive and player II protective. Thus, to verify properties of competitive games, it is sufficient to present proof for the pertinent case(s) of one player protective and the other vindictive. Finally, it is to be noted that competitive players can have different preferred sequences in the sense of different combinations of strategies being associated with outcomes that have the same value. However, this causes no difficulties in derivations since the preferred sequences are the same with respect to the values of the outcomes.

The following three theorems contain the verification that is not evident from the properties of the procedures for determining P_I, P_{II}, P'_I, and P'_{II}.

Theorem 1. The procedure of marking payoffs (in his matrix) for a player until the first time that two or fewer rows contain marked values in all columns guarantees that occurrence of a marked value can be assured with a probability of at least 1/2.

Proof: First note that continued marking ultimately results in this situation. When one row becomes fully marked, the probability is unity that some one of the marked values can be assured by the player.

Next, suppose that a pair of rows is needed. When two mixed strategies $p_1, \ldots, p_r$ and $q_1, \ldots, q_s$ are used (pure strategies are special cases, and there are r rows and s columns), the probability of the marked subset is

$$\sum_{i=1}^{r} p_i Q_i ,$$

where Q_i is the sum of the q's for columns that have marked payoffs in the i-th row. The largest value of this probability that the player can assure (by choice of $p_1, \ldots, p_r$) is

$$G = \min_{q_1, \ldots, q_s} (\max_i Q_i) .$$

Let i(1) and i(2) denote the two rows that together contain marked payoffs in all columns. For any minimizing set of q's, both $Q_{i(1)}$ and $Q_{i(2)}$ are at most G, so that

$$2G \geq Q_{i(1)} + Q_{i(2)} \geq 1 ,$$

and a probability of at least 1/2 can be assured. This probability can exceed 1/2 but is exactly 1/2 when the unmarked payoffs are such that two columns contain unmarked payoffs in all rows (since analogously, the set of unmarked payoffs can be assured with a probability of at least 1/2). It is also exactly 1/2 when there are two columns that have an unmarked payoff in row i(1) or row i(2) and are such that no row of the matrix has payoffs marked in both columns.

Theorem 2. A lower bound on the probability that a player can assure one of a specified subset of outcomes, and corresponding optimum strategies, can be determined by solution of a zero-sum game with an expected value basis. The payoff matrix for this game has ones at the positions that correspond to the (pure) strategy combinations for the subset of outcomes, and zeroes elsewhere.

Proof: Let each player use an arbitrary mixed strategy (a pure strategy occurs as a special case). The expression for the expected payoff of the zero-sum game is also the expression for the probability that some one of the outcomes in the specified subset occurs.

Theorem 3. When protective player I (II) has a fully marked row in his matrix, but vindictive player II (I) does not have a fully marked column in this matrix, $P_I' = P_I$ $(P_{II}' = P_{II})$; also, player II (I) can assure that player I (II) receives at most P_I' (P_{II}') with a probability greater than 1/2. Likewise, when vindictive player I (II) has a fully marked column in the matrix for protective player II (I), but player II (I) does not have a fully marked row in this matrix, $P_{II}' = P_{II}$ $(P_I' = P_I)$; also, player I (II) can assure himself at least P_I (P_{II}) with a probability greater than 1/2.

Proof: Consider the outcome that corresponds to the last payoff marked for protective player I (II) and the outcomes that do not correspond to marked payoffs for player I (II). This set of outcomes can be assured with probability greater than 1/2 by vindictive player II (I). Otherwise, player I (II) would have terminated his marking procedure earlier. The payoffs for player I (II) in this set of outcomes are at most equal to P'_{I} (P'_{II}), with equality holding for the outcome corresponding to the last payoff marked for player I (II). This follows from the development of preferred sequences for competitive games. Also, since player I (II) has a fully marked row in his matrix, player II (I) cannot assure that player I (II) receives any payoff less than P'_{I} with nonzero probability. Thus, $P_{I} = P'_{I}$ ($P_{II} = P'_{II}$) and player II (I) can assure, with a probability greater than 1/2, that player I (II) receives at most P'_{I} (P'_{II}).

A similar verification can be given for the case of vindictive player I (II) having a fully marked column and protective player II (I) not having a fully marked row.