APPROXIMATING TAIL PROBABILITIES OF NONCENTRAL DISTRIBUTIONS

by

Suojin Wang and H.L. Gray
Southern Methodist University

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Suojin Wang and H. L. Gray
Department of Statistical Science
Southern Methodist University

Abstract

Gray and Wang's (1991) general method for approximating tail probabilities is applied to the cases of noncentral χ^2, F and t distributions. The validity of such applications is established. The resulting approximations are easy to compute. Numerical results show the great accuracy of the approximations for all three most commonly used noncentral distributions.

Keywords: $G_n^{(m)}$-transform, noncentral χ^2, noncentral F, noncentral t, tail probability.

1. Introduction

Calculating the tail probabilities of noncentral distributions such as noncentral χ^2, F and t is a major step in many statistical applications. However, these distributions do not have a simple closed form for the cumulative distribution functions (CDFs) which may be expressed in the form of infinite series only. Many methods have been proposed for approximating the CDFs of these noncentral distributions by truncating the infinite series or using central distributions as approximations. These methods are designed for each specific distribution.

Taking a totally different approach from the viewpoint of the Generalized Jackknife, Gray and Wang (1991) have introduced a general method referred to as the $G_n^{(m)}$-transformation for finding functions which are easy to evaluate and give very good approximations to tail probabilities. The transformation is based only on a general class of differential equations that include the specified density function in the solution set. As a result this method applies to a broad class of distributions.

Since the densities of noncentral distributions are themselves in the form of infinite series, it is far from obvious how the conditions for the $G_n^{(m)}$-transformation are met in these cases. In this note we show that the three most commonly used noncentral distributions (noncentral χ^2, F and t) all satisfy the conditions so that the $G_n^{(m)}$-transformation provides a good alternative in approximating the tail probabilties of these distributions. Its high degree of accuracy even in the extreme tails is maintained, as will be shown in numerical examples.

We first briefly review the $G_n^{(m)}$-transformation. Let f be a density function and assume that we wish to approximate the tail probability

$$G(x) = \int_x^{\infty} f(t)\, dt \,. \tag{1}$$

Let

$$U_k(x) = x^{\ell_k} \sum_{i=0}^{\infty} \frac{\alpha_{k,i}}{x^i} \,,$$

where $\alpha_{k,0} \neq 0$ and ℓ_k is an integer with $\ell_k \leq k$. Suppose that f(x) satisfies the following differential equation

$$U_m(x)\, f^{(m)}(x) + U_{m-1}(x)\, f^{(m-1)}(x) + \ldots + U_1(x)\, f'(x) - f(x) = 0 \,, \tag{2}$$

for some collection of U_k's, where m is assumed to be the smallest possible integer such that (2) holds. Using the idea of the Generalized Jackknife, Gray and Wang (1991) define the $G_n^{(m)}$-transformation approximating G(x) in (1) as

$$G_n^{(m)}[f(x); a_{ij}(x)] = \frac{\begin{vmatrix} 0 & f(x) & \cdots & f^{(N-1)}(x) \\ a_{11}(x) & a_{12}(x) & \cdots & a_{1,N+1}(x) \\ \vdots & \vdots & & \vdots \\ a_{N,1}(x) & a_{N,2}(x) & \cdots & a_{N,N+1}(x) \end{vmatrix}}{\begin{vmatrix} 1 & 0 & \cdots & 0 \\ a_{11}(x) & a_{12}(x) & \cdots & a_{1,N+1}(x) \\ \vdots & \vdots & & \vdots \\ a_{N,1}(x) & a_{N,2}(x) & \cdots & a_{N,N+1}(x) \end{vmatrix}} \,, \tag{3}$$

where

$$a_{ij}(x) = \begin{cases} \left\{x^{\ell_1-i+1}f(x)\right\}^{(j-1)}, & i = 1, \ldots, n, \\ \left\{x^{\ell_2-i+n+1}f'(x)\right\}^{(j-1)}, & i = n+1, \ldots, 2n, \\ \left\{x^{\ell_m-i+(m-1)n+1}f^{(m-1)}(x)\right\}^{(j-1)}, & i = (m-1)n+1, \ldots, N, \end{cases}$$

for $j = 1, \ldots, N+1$, and $N = mn$.

As shown in Gray and Wang (1991), most commonly used simple continuous distributions including those in the Pearson family satisfy (2) for $m = 1$. Furthermore, it has been shown that under mild regularity conditions,

$$\lim_{n\to\infty} G_n^{(m)}[f(x); a_{ij}(x)] = G(x) . \tag{4}$$

Gray and Wang (1991) have examplified the extraordinary accuracy of the $G_n^{(m)}$-transform for n as small as 2 or 3.

2. Noncentral χ^2 distribution

The first application of the $G_n^{(m)}$-transformation we consider is to the noncentral chi-squared distribution $\chi_k'^2(\lambda)$, where k is the degree of freedom and λ is the noncentrality parameter. Cox and Reid (1987) obtain an approximation to the distribution of $\chi_k'^2(\lambda)$ by adding a small perturbation to the corresponding central χ_k^2 random variable when λ is small relative to k. Cohen (1988) and Posten (1989) have developed algorithms for the distribution of $\chi_k'^2(\lambda)$. These algorithms require evaluating either the central χ_k^2 distribution function or the distribution functions of $\chi_i'^2(\lambda)$ for the lowest degrees of freedom $i = 1$, 2 and 3.

The density of $\chi_k'^2(\lambda)$ can be expressed as (Johnson and Kotz, 1970, p. 132)

$$f(x) = \frac{e^{-(\lambda+x)/2}}{2^{k/2}} \sum_{r=0}^{\infty} \frac{\left(\frac{\lambda}{4}\right)^r x^{k/2+r-1}}{r!\,\Gamma(k/2+r)}, \; x > 0. \tag{5}$$

Now note that

$$f(x) = e^{-\lambda/2}\, g_k(x)\ {}_0F_1\left(-; \tfrac{k}{2}; \tfrac{\lambda x}{4}\right), \tag{6}$$

where $g_k(x) = e^{-x/2}\, x^{k/2\,-\,1}/\left(\Gamma(\frac{k}{2})\, 2^{k/2}\right)$ is the density of χ^2_k,

$${}_0F_1(-; b; z) = \sum_{r=0}^{\infty} \frac{z^r}{r!(b)_r}$$

is a hypergeometric function (Rainville, 1960, p. 74) and

$$(b)_r = \begin{cases} 1, & r = 0, \\ b(b+1)\ldots(b+r-1), & r = 1, 2, \ldots. \end{cases}$$

Let $u = {}_0F_1(-; b; z)$. It has been shown (Rainville, 1960, p. 109) that

$$z\frac{d^2u}{dz^2} + b\frac{du}{dz} - u = 0\,. \tag{7}$$

Let $b = \frac{k}{2}$ and $z = \frac{\lambda x}{4}$. It follows from (7) that

$$\frac{4x}{\lambda}\frac{d^2u}{dx^2} + \frac{2k}{\lambda}\frac{du}{dx} - u = 0\,. \tag{8}$$

Substituting $u = e^{\lambda/2} f(x)/g_k(x)$ into (8) and after some algebra we have

$$x\, f''(x) + \left(x + (2-\tfrac{k}{2})\right) f'(x) + \frac{x+4-k-\lambda}{4}\, f(x) = 0\,, \tag{9}$$

where we used the fact that $g_k'(x) = \left(-\frac{1}{2} + (\frac{k}{2} - 1)/x\right) g_k(x)$. We have therefore shown that f(x) satisfies (2) with $m = 2$ and $\ell_1 = \ell_2 = 0$. Hence the application of the $G_n^{(m)}$-transform, denoted by $G_n^{(2)}[f(x); a_{ij}(x)]$, is straightforward as long as $f^{(r)}(x)$, $r = 0, 1, \ldots, 2n+1$, are obtained.

Taking r-th derivative of (9) and after some algebra we have the recursive formula

$$f^{(r+2)}(x) = \left\{\left(\frac{k}{2} - 2 - r - x\right) f^{(r+1)}(x) + \left(\frac{k+\lambda-x}{4} - 1 - r\right) f^{(r)}(x) - \frac{r}{4} f^{(r-1)}(x)\right\}/x\,, \tag{10}$$

for $r = 1, 2, \ldots$. From (9) it is easily seen that equation (10) is also valid for $r = 0$ if we define $f^{(-1)}(x) \equiv 0$. Thus we need only to evaluate $f(x)$ and $f'(x)$. But from (6) we have

$$f'(x) = \left(-\frac{1}{2} + \frac{k-2}{2x}\right) f(x) + \frac{\lambda}{2k} e^{-\lambda/2} g_k(x)\, {}_0F_1\left(-; \frac{k+2}{2}; \frac{\lambda x}{4}\right). \tag{11}$$

By (6) and (11), it is sufficient to compute a hypergeometric function of the form

$${}_0F_1(-; b; z) = \sum_{r=0}^{\infty} \frac{z^r}{r!(b)_r} = \sum_{r=0}^{\infty} c_r, \tag{12}$$

where $c_0 = 1$, $c_r = \frac{z}{r(b+r-1)} c_{r-1}$, $r \geq 1$. The series in (12) converges as quickly as $\sum_{r=1}^{\infty} \frac{z^r}{(r!)^2}$ asymptotically. Note that Levine and Sidi's (1981) d-transformation is useful to accelerate the convergence of the series in (12).

To compute $G_n^{(2)}[f(x); a_{ij}(x)]$ in (3) a subroutine for computing the determinant of a matrix is normally required. But when $n = 1$ it is easily derived that in the noncentral χ^2 case,

$$G_1^{(2)}[f(x); a_{ij}(x)] = \frac{A(x)}{B(x)}\,, \tag{13}$$

where

$$\begin{aligned} A(x) = &\left[-\frac{x}{4}\left(x - \frac{3k}{2} - \lambda + 6\right) - \frac{k-\lambda-4}{4}\left(\frac{k}{2} - 3\right)\right] f^3(x) \\ &+ \left[\frac{x}{4}(-5x+\lambda+5k-20) - \left(\frac{k}{2} - 2\right)\left(\frac{k}{2} - 3\right)\right] f^2(x) f'(x) \\ &+ \left[-2x^2 + (k-4)x\right] f(x)\left(f'(x)\right)^2 - x^2\left(f'(x)\right)^3, \end{aligned}$$

$$B(x) = \left[\frac{x}{4}(-x+k+\lambda-4)-\frac{k}{2}+2\right]\left(f'(x)\right)^2+\left[\frac{x}{4}\left(-x+\frac{3k}{2}+\lambda-6\right)+\frac{1}{8}(k+\lambda-4)(2-k)\right]f(x)f'(x)$$

$$-\frac{1}{16}(-x+k+\lambda-4)^2 f^2(x).$$

Table 1 provides some selected values of $G_n^{(2)}$. Rapid convergence of $G_n^{(2)}$ as n increases is evident; great accuracy is obtained for n as small as 2. It is seen that the approximation is particularly useful for extreme tails. We used a finite sum to approximate the series in (12), and therefore f(x) and $f'(x)$. In Table 1, 15-40 terms were needed depending on different parameters to get at least 12 significant digits correct for f(x) and $f'(x)$. Applying Levine and Sidi's (1981) acceleration method improves this by at least 3 more significant digits. However the quick convergence in (12) makes it generally unnecessary to use Levine and Sidi's method. In this and the following applications, we used the IMSL subroutine LINV3F to calculate determinants. Any other efficient algorithms can be used for this purpose.

3. Noncentral F distribution

We now consider the application of the $G_n^{(m)}$-transform to noncentral F distributions. Despite the different structures of the noncentral F and noncentral χ^2 distributions, a similar procedure can be developed. Like in the noncentral χ^2 case our approach is different from those usual ones which use central F distributions as approximations; see Hirotsu (1979). The density of a noncentral F distribution with degrees of freedom k and p and noncentrality parameter λ (denoted by $F'_{k,p}(\lambda)$) is (Johnson and Kotz, 1970, p. 191)

$$g(y) = \frac{e^{-\lambda/2}(\frac{k}{p})^{k/2}\, y^{k/2-1}}{\Gamma(\frac{p}{2})(1+\frac{k}{p}\,y)^{(k+p)/2}} \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{k+p}{2}+r\right)}{r!\,\Gamma(\frac{k}{2}+r)} \left(\frac{\frac{\lambda k}{2p}\,y}{1+\frac{k}{p}\,y}\right)^r$$

$$= e^{-\lambda/2} h_{k,p}(y)\ {}_1F_1\left(\frac{k+p}{2};\ \frac{k}{2};\ \frac{\lambda}{2}\,\frac{ky}{p+ky}\right), \tag{14}$$

where

$$h_{k,p}(y) = \frac{\Gamma\left(\frac{k+p}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{p}{2}\right)} \frac{\left(\frac{k}{p}\right)^{k/2} y^{k/2 - 1}}{\left(1+\frac{k}{p}y\right)^{(k+p)/2}}$$

is the density for a central F distribution with degrees of freedom k and p, and

$$_1F_1(a;\, b;\, z) = \sum_{r=0}^{\infty} \frac{(a)_r}{r!(b)_r} z^r$$

is a hypergeometric function. Since

$$G(y) = \int_y^{\infty} g(t)dt = \int_x^{\infty} g\left(\frac{p}{k}s\right)\frac{p}{k}\, ds = \int_x^{\infty} f(s)\, ds\,, \tag{15}$$

where $x = \frac{k}{p}y$,

$$f(x) = g\left(\frac{p}{k}x\right)\frac{p}{k} = e^{-\lambda/2}\, h_{k,p}\left(\frac{p}{k}x\right) {}_1F_1\left(\frac{k+p}{2};\ \frac{k}{2};\ \frac{\lambda x}{2(1+x)}\right), \tag{16}$$

we need only to consider the transformed density f(x). Let $u = {}_1F_1(a;\, b;\, z)$. Using the fact that (Rainville, 1960, p. 124)

$$z\frac{d^2u}{dz^2} + (b-z)\frac{du}{dz} - a\, u = 0 \tag{17}$$

and after lengthy algebra similar to those in Section 2 we can obtain that

$$f''(x) = \left\{\frac{k-4}{2x} - \frac{k+2p+4}{2(1+x)} + \frac{\lambda}{2(1+x)^2}\right\} f'(x) + \frac{p+2}{4}\left\{(k+\lambda-4)\left(\frac{1}{x} - \frac{1}{1+x}\right) + \frac{k+p+\lambda}{(1+x)^2}\right\} f(x)\,. \tag{18}$$

Comparing (18) with (2), it is readily seen that $m = 2$, $\ell_1 = 1$ and $\ell_2 = 2$.

A recursive formula for higher order derivatives of f(x) can be easily obtained by taking derivatives on both sides of (18) and by the fact that

$$\left(h_1(x)\, h_2(x)\right)^{(i)} = \sum_{j=0}^{i} \binom{i}{j} h_1^{(j)}(x)\, h_2^{(i-j)}(x), \quad i = 1, 2, \ldots.$$

It is therefore only necessary to evaluate f(x) and f′(x). But

$$f'(x) = \left[\frac{k-2}{2x} - \frac{k+p}{2(1+x)}\right] f(x) + \frac{e^{-\lambda/2}(k+p)\lambda\, x^{k/2\,-\,1}}{2k\Gamma(p/2)(1+x)^{(k+p)/2\,+\,2}}\, {}_1F_1\!\left(\frac{k+p+2}{2}; \frac{k+2}{2}; \frac{\lambda x}{2(1+x)}\right), \tag{19}$$

so that the essential part is to compute ${}_1F_1(a; b; z)$. Similar to (12),

$$ {}_1F_1(a; b; z) = \sum_{r=0}^{\infty} d_r\,, \tag{20}$$

where $d_0 = 1$, $d_r = \frac{(a+r-1)z}{r(b+r-1)}\, d_{r-1}$, $r = 1, 2, \ldots$. Therefore ${}_1F_1(a; b; z)$ converges as fast as $\sum_{r=1}^{\infty} \frac{z^r}{r!}$ asymptotically.

Selected values of the $G_n^{(2)}$-transform (n = 1, 2, 3) are given in Table 2. It is clearly seen that n = 2 or 3 is generally sufficient in most applications. In Table 2, up to 50 terms in (20) were used to approximate f(x) in (16) and f′(x) in (19) to at least 12 significant digits. The last column is the Edgeworth-series approximation by Mudholkar, Chaubey and Lin (1976). Table 1 shows that for n as small as 3 the general method $G_n^{(2)}$ is more accurate in the tail than the Edgeworth-series approximation designed only for the noncentral F.

4. Noncentral t distribution

This section concerns with noncentral t distributions. Kraemer and Paik (1979) proposed approximations based on central t distributions assuming that the noncentrality parameter is small relative to the degree of freedom. As we have mentioned earlier, our method does not require such assumption on the noncentrality parameter. The density of a noncentral t distribution with k degrees of freedom and noncentrality parameter λ (denoted by $t'_k(\lambda)$) is (Johnson and Kotz, 1970, p. 205)

$$g(y) = \frac{e^{-\lambda^2/2}\left(\frac{k}{k+y^2}\right)^{(k+1)/2}}{\sqrt{\pi k}\;\Gamma(\frac{k}{2})} \quad \sum_{r=0}^{\infty} \frac{\Gamma\!\left(\frac{k+r+1}{2}\right)}{r!} \left(\frac{\sqrt{2}\,\lambda\, y}{\sqrt{k+y^2}}\right)^{r}. \tag{21}$$

Let $x = 1 + y^2/k$. It follows that

$$G(y) = \int_y^\infty g(t)dt = \int_x^\infty g\left(\{k(s-1)\}^{1/2}\right) \frac{k}{2\{k(s-1)\}^{1/2}} ds = \int_x^\infty f(s)\, ds, \tag{22}$$

where

$$f(x) = g\left(\{k(x-1)\}^{1/2}\right) \frac{k}{2\{k(x-1)\}^{1/2}}$$

$$= \frac{e^{-\lambda^2/2}}{2\sqrt{\pi}\,\Gamma(\frac{k}{2})(x-1)^{1/2} x^{(k+1)/2}} \left\{ \sum_{\substack{r=2j \\ j\geq 0}} + \sum_{\substack{r=2j+1 \\ j\geq 0}} \right\} \frac{\Gamma\left(\frac{k+r+1}{2}\right)}{r!} \left(\sqrt{2}\lambda\left(1-\frac{1}{x}\right)\right)^r$$

$$= \frac{e^{-\lambda^2/2}\,\Gamma\left(\frac{k+1}{2}\right)}{2\sqrt{\pi}\,\Gamma(\frac{k}{2})(x-1)^{1/2} x^{(k+1)/2}}\, {}_1F_1\left(\frac{k+1}{2};\ \frac{1}{2};\ \frac{\lambda^2}{2}\left(1-\frac{1}{x}\right)\right)$$

$$+ \frac{\lambda e^{-\lambda^2/2}\Gamma\left(\frac{k}{2}+1\right)}{\sqrt{2\pi}\,\Gamma\left(\frac{k}{2}\right) x^{k/2+1}}\, {}_1F_1\left(\frac{k}{2}+1;\ \frac{3}{2};\ \frac{\lambda^2}{2}\left(1-\frac{1}{x}\right)\right) \tag{23}$$

$$= f_1(x) + f_2(x)\,, \tag{24}$$

f_1 and f_2 are defined by the first and second term in (23), respectively. It is easily derived that

$$f''(x) = \left\{\frac{-3}{2(x-1)} - \frac{2k+5}{2x} + \frac{\lambda^2}{2x^2}\right\} f'(x)$$

$$+ \left\{\frac{(k+2)(\lambda^2-3)}{4}\left(\frac{1}{x-1} - \frac{1}{x}\right) - \frac{(k+2)(k+1+\lambda^2)}{4x^2}\right\} f(x), \tag{25}$$

since using (17) and the same technique as that used in Sections 2 and 3 we can show that both $f_1(x)$ and $f_2(x)$ satisfy the differential equation (25). Comparing (25) with (2) we have $m = 2$, $\ell_1 = 1$ and $\ell_2 = 2$. As in the case of the noncentral χ^2 (or F), a recursive formula for $f^{(i)}(x)$ $(i = 2, 3, \ldots)$ in terms

of lower order derivatives can be easily obtained from (25). Thus we need only the evaluation of f(x) and f′(x) to calculate all required derivatives. It is easily obtained from (24) that

$$f'(x) = \left(\frac{-1}{2(x-1)} - \frac{k+1}{2x}\right) f_1(x) + \frac{e^{-\lambda^2/2}\,\Gamma\left(\frac{k+1}{2}\right)\lambda^2(k+1)}{4\sqrt{\pi}\,\Gamma(\frac{k}{2})(x-1)^{1/2}x^{(k+5)/2}}\,{}_1F_1\left(\frac{k+3}{2};\ \frac{3}{2};\ \frac{\lambda^2}{2}\left(1-\frac{1}{x}\right)\right)$$

$$-\frac{k+2}{2x} f_2(x) + \frac{\lambda^3 e^{-\lambda^2/2}\,\Gamma\left(\frac{k}{2}+1\right)(k+2)}{6\sqrt{2\pi}\,\Gamma(\frac{k}{2})\,x^{k/2+3}}\,{}_1F_1\left(\frac{k}{2}+2;\ \frac{5}{2};\ \frac{\lambda^2}{2}\left(1-\frac{1}{x}\right)\right). \tag{26}$$

It follows from (23) that evaluation of hypergeometric functions of type ${}_1F_1(a;\, b;\, z)$ is a main step which has been addressed in Section 3.

Some numerical results are given in Table 3 to show the quick convergence of the $G_n^{(2)}$-transform. Like in the previous cases, n = 2 or 3 is usually sufficiently accurate.

5. Conclusion

In this paper, we have considered the applications of the general method of $G_n^{(m)}$-transform for tail probabilities to the three most commonly used noncentral distributions. The validity of such applications was verified and accurate numerical results were given.

Except for calculating the density and its derivatives, the $G_n^{(m)}$-transform does not use any particular properties of a specified distribution, making it a very general method. The method is also easily implemented in practice. A short self-explanatory FORTRAN subroutine for the $G_n^{(m)}$-transform is given in the appendix.

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APPENDIX: A SUBROUTINE FOR THE $G_n^{(m)}$-TRANSFORM

```
      SUBROUTINE GMNTRANS (M, N, L, X, FDVEC, GMN)
C
C        THIS SUBROUTINE CALCULATES THE GMN-TRANSFORM.
C        INPUT: M, N, L, X, FDVEC;   OUTPUT: GMN(1),...,GMN(N).
C        M IS DEFINED IN EQUATION (2);
C        N IS THE MAXIMUM ORDER OF THE TRANSFORM;
C        L = (L(1),...,L(M)) IS DEFINED IN EQUATION (2) (SET
C             L(I)=I, IF L(I) UNKNOWN);
C        FDVEC = (FDVEC(1),...,FDVEC(M*N+M)), WHERE FDVEC(I)
C             IS THE (I-1)TH DERIVATIVE OF THE DENSITY AT X.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION FDVEC(31), L(3), A1(30,30), A2(30,30), GMN(15)
      DIMENSION WKA(30), JVEC(30,30), PDM(30,31), B(1)
      IJOB = 4
      DO 100 NI=1, N
         NTL = 1+M*NI
         A1(1,1) = 1.D0
         A2(1,1) = 0.D0
         DO 10 J = 2, NTL
            A1(1,J) = 0.D0
10          A2(1,J) = FDVEC(J-1)
         DO 40 J = 1, NTL
            JVEC(1,J) = 1
            JVEC(J,J) = 1
            IF (J.EQ.1.OR.J.EQ.2) GO TO 40
            IF (J.EQ.3) GO TO 30
            JMD2 = J/2-1
            DO 20 K = 1, JMD2
               JVEC(K+1,J) = JVEC(K,J)*(J-K)/K
20             JVEC(J-K,J) = JVEC(K+1,J)
30          JD = (J-1)/2
            IF (J-1.EQ.JD*2) JVEC(JD+1,J) = JVEC(JD,J)*(JD+1)/JD
40       CONTINUE
         DO 60 K = 0, M-1
         DO 60 I = 1, NI
            POW = L(K+1)-I+1
            PDM(I+K*NI+1,1) = X**POW
            DO 60 J = 1, NTL
               PDM(I+K*NI+1,J+1) = PDM(I+K*NI+1,J)*(POW-J+1)/X
               A1(I+K*NI+1,J) = 0.D0
               DO 50 I1 = 1, J
50                A1(I+K*NI+1,J) = A1(I+K*NI+1,J)+JVEC(I1,J)
     $                  *PDM(I+K*NI+1,I1)*FDVEC(J-I1+K+1)
               A2(I+K*NI+1,J) = A1(I+K*NI+1,J)
60       CONTINUE
         D1 = 10.D0
         CALL LINV3F(A1, B, IJOB, NTL, 30, D1, D2, WKA, IER)
C          HERE WE USE IMSL SUBROUTINE LINV3F TO CALCULATE
C          THE DETERMINANT OF A MATRIX.
C          OTHER EFFICIENT SUBROUTINES CAN ALSO BE USED HERE.
         D3 = 10.D0
         CALL LINV3F(A2, B, IJOB, NTL, 30, D3, D4, WKA, IER)
100   GMN(NI) = D3/D1*2.D0**(D4-D2)
      RETURN
      END
```

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Table 1. Relative error = R[approx.] = { | approx. − true |/true}

of approximations to the upper tail of $\chi_k'^2(\lambda)$.

k	λ	x	true value	$R[G_1^{(2)}]$	$R[G_2^{(2)}]$	$R[G_3^{(2)}]$
5	1	8	.2466	1.1(2)*	1.5(4)	1.1(4)
		13	.07332	2.8(3)	1.5(5)	4.5(7)
		21	.003156	4.6(4)	5.1(7)	3.8(9)
5	10	20	.2189	6.2(2)	7.9(4)	8.3(6)
		26	.07497	1.9(2)	1.5(4)	8.4(7)
		43	.001521	2.3(3)	5.4(6)	5.5(9)
25	10	44	.1674	8.1(2)	8.0(3)	6.6(4)
		50	.06824	3.2(2)	1.8(3)	9.9(5)
		70	.001337	3.7(3)	4.8(5)	7.8(7)
25	25	62	.1616	9.5(2)	9.4(3)	7.4(4)
		70	.06159	3.7(2)	2.0(3)	1.0(4)
		94	.001295	4.7(3)	6.4(5)	1.1(6)

* $1.1(2) = 1.1 \times 10^{-2}$

Table 2. Relative error = R[approx.] = { | approx. − true | /true} of approximations to the upper tail of $F'_{k,p}(\lambda)$.

k	p	λ	y	true value	$R[G_2^{(2)}]$	$R[G_3^{(2)}]$	Edgeworth
2	4	2	5	.2003	2.6(4)*	1.4(7)	1.7(5)
			10	.07906	2.1(5)	4.2(9)	2.9(3)
			33	.01065	2.1(7)	5.1(12)	6.1(3)
2	4	15	20	.2135	9.1(4)	4.9(7)	4.2(4)
			42	.06946	3.3(5)	5.5(9)	8.1(3)
			106	.01353	6.7(7)	2.3(11)	7.6(3)
2	20	10	10	.1817	1.5(2)	1.3(5)	1.5(3)
			15	.05327	1.7(3)	6.4(7)	3.0(3)
			25	.005622	1.2(4)	5.3(8)	5.9(3)
20	20	20	3	.1775	1.7(2)	8.1(4)	4.5(4)
			4	.05722	2.2(3)	5.3(5)	6.6(5)
			10	.002323	1.0(5)	3.0(8)	1.2(2)

* 2.6(4) = 2.6×10^{-4}

Table 3. Relative error = R[approx.] = { | approx. − true |/true} of approximations to the upper tail of $t'_k(\lambda)$.

k	λ	y	true value	$R[G_1^{(2)}]$	$R[G_2^{(2)}]$	$R[G_3^{(2)}]$
3	1	2	.2564	2.5(2)*	1.1(3)	1.2(4)
		4	.06390	1.3(3)	3.8(6)	5.7(8)
		8	.01012	7.3(5)	1.0(8)	1.9(11)
3	7	12	.2087	2.9(2)	5.4(4)	7.1(4)
		19	.06397	3.4(3)	3.6(5)	1.9(5)
		37	.009575	2.0(4)	1.4(8)	1.9(9)
10	7	10	.1263	5.2(2)	1.7(3)	3.5(5)
		12	.04187	1.6(2)	2.2(4)	2.1(6)
		17	.003113	2.5(3)	6.9(6)	1.3(8)

* 2.5(2) = 2.5×10^{-2}