CORRELATED RANDOM WALKS WITH STAY

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ABSTRACT

A random walk describes the movement of a particle in discrete time, with the direction and the distance traversed in one step being governed by a probability distribution. In a correlated random walk (CRW) the movement follows a Markov chain and induces correlation in the state of the walk at various epochs. Then, the walk can be modelled as a bivariate Markov chain with the location of the particle and the direction of movement as the two variables. In such random walks, normally, the particle is not allowed to stay at one location from one step to the next. In this paper we derive explicit results for the following characteristics of the CRW when it is allowed to stay at the same location, directly from its transition probability matrix: (i) equilibrium solution and the first passage probabilities for the CRW restricted on one side, and (ii) equilibrium solution and first passage characteristics for the CRW restricted on both sides (i.e., with finite state space).

Key words: Correlated random walk, Markov chain, equilibrium solution, first passage time.

1. INTRODUCTION

A random walk describes the movement of a particle in discrete time, with the direction and the distance traversed in one step being governed by a probability distribution. In a correlated random walk (CRW) the probability distribution at successive epochs of the walk are dependent on the preceding steps in a Markovian strucutre. Consequently the walk itself becomes a correlated process. Starting with Gillis (1955) and Mohan (1955) several authors have investigated problems related to CRW in one and more dimensions, and its special cases such as the symmetric CRW which has the same probability for proceeding in the same direction in successive epochs. All these investigations have used the classical approach of difference equations and generating functions. Except for Nain and Sen (1979a), the papers by Gupta (1958), Seth (1963), Jain (1971, 1973), Darroch and Whitford (1972), Proudfoot and Lampard (1972), Nain and Sen (1979b, 1980), Renshaw and Henderson (1981), Henderson et al. (1983, 1984), Bender and Richmond (1984) and Roerdink (1985) have considered CRW in which the particle is not allowed to stay at the same location from one epoch to the next. Also in most of these investigations first passage problems are of primary interest. In the lone investigation of unrestricted CRW with stay, Nain and Sen (1979a) consider characteristics such as first passage to a specific state, return to the origin, passage to a state a given number of times, and transitions crossing the x-axis and rebounding from it. The results are mostly in terms of generating functions. In this investigation we propose to concentrate on the equilibrium distribution of the walk when restrictions are imposed on the state space on one or both sides and some characteristics such as the probability of first return into a set of

contiguous states and first passage probabilites and mean passage times with respect to specified states. All results are either in explicit or in easily computable expressions.

The investigation is facilitated by representing CRW by a bivariate process $\{(W_n,U_n),\ n = 0,1,2,\ldots\}$, where W_n is the location of the particle after n steps and U_n is the nature of the step signified by -1, for a move to the left, 0 for stay in the same state, and +1 for a move to the right. We consider the walk restricted to the non-negative integers, thus resulting in a product space for the walk with factors

$$W = (\ 0,\ 1,\ 2,\ \ldots\ )$$

$$U = (-1,\ 0,\ +1)$$

The process $\{U_n\}$ determining the direction of the walk is a Markov chain with transition probability matrix

$$P_U = \begin{array}{c} \\ +1 \\ 0 \\ -1 \end{array} \begin{array}{c} \begin{array}{ccc} +1 & 0 & -1 \end{array} \\ \begin{bmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ g & h & 1-g-h \end{bmatrix} \end{array} \tag{1.1}$$

$$0 \leq a,\ b,\ c,\ d,\ g,\ h \leq 1\ .$$

We also assume that the elements are such that this Markov chain is irreducible and aperiodic. As $n \to \infty$, $\{U_n,\ n = 0,\ 1,\ 2,\ \ldots\ \}$ has the following properties.

Let $\underline{\pi} = (\pi_{+1}, \pi_0, \pi_{-1})$ be the limiting distribution of U_n. Using well known properties of Markov chains ($\underline{\pi}P_U = \underline{\pi}$ and $\underline{\pi}\ \underline{e} = 1$), we get

$$\pi_{+1} = \frac{hc + (1-d)g}{(1-a)(1-d)-bc}\ \pi_{-1} \tag{1.2}$$

$$\pi_0 = \frac{bg + (1-a)h}{(1-d)(1-a)-bc}\ \pi_{-1}$$

leading to the result

$$\begin{aligned} \pi_{+1} &= [hc + (1-d)g]/\omega \\ \pi_0 &= [bg + (1-a)h]/\omega \\ \pi_{-1} &= [(1-a)(1-d) - bc]/\omega \end{aligned} \tag{1.3}$$

where

$$\omega = (1-d)(1-a+g) + h(1-a+c) + b(g-c).$$

Assuming that U_0 has the distribution given by (1.3), the serial correlation with lag 1 of the process $\{U_n\}$ can be obtained as

$$\rho = \left[(2a+b-1)\pi_{+1} - (2g+h-1)\pi_{-1} - [E(U)]^2\right]/V(U)$$

where

$$\begin{aligned} E(U) &= \pi_{+1} - \pi_{-1} \\ V(U) &= \pi_{+1} + \pi_{-1} - [E(U)]^2 \end{aligned} \tag{1.4}$$

The transition probability matrix of the CRW can be now given as follows.

$$
P = \begin{array}{c}
\\ (0,0) \\ (0,-1) \\ (1,+1) \\ (1,0) \\ (1,-1) \\ (2,+1) \\ (2,0) \\ (2,-1) \\ \vdots
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc}
(0,0) & (0,-1) & (1,+1) & (1,0) & (1,-1) & (2,+1) & (2,0) & (2,-1) & (3,+1) & (3,0) & (3,-1) & \cdots
\end{array} \\
\left[\begin{array}{cc|ccc|ccc|ccc|c}
d & 0 & 1-d & 0 & 0 & & & & & & & \\
h & 0 & 1-h & 0 & 0 & & 0 & & & 0 & & \cdots \\
\hline
0 & 1-a-b & 0 & b & 0 & a & 0 & 0 & & & & \\
0 & 1-c-d & 0 & d & 0 & c & 0 & 0 & & 0 & & \cdots \\
0 & 1-g-h & 0 & h & 0 & g & 0 & 0 & & & & \\
\hline
 & & 0 & 0 & 1-a-b & 0 & b & 0 & a & 0 & 0 & \\
 & 0 & 0 & 0 & 1-c-d & 0 & d & 0 & c & 0 & 0 & \cdots \\
 & & 0 & 0 & 1-g-h & 0 & h & 0 & g & 0 & 0 & \\
\hline
\vdots & \vdots & & \vdots & & & \vdots & & & \vdots & & \\
\end{array}\right]
\end{array}
$$

$$
= \begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots
\end{array}
\begin{array}{c}
\begin{array}{ccccc}
0 & 1 & 2 & 3 & \cdots
\end{array} \\
\begin{bmatrix}
A & B & & & \\
C & Z & Y & & \\
 & X & Z & Y & \\
 & & X & Z & Y \\
 & & & & \ddots
\end{bmatrix}
\end{array}
\tag{1.5}
$$

where

$$
A = \begin{bmatrix} d & 0 \\ h & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1-d & 0 & 0 \\ 1-h & 0 & 0 \end{bmatrix}
$$

$$C = \begin{bmatrix} 0 & 1-a-b \\ 0 & 1-c-d \\ 0 & 1-g-h \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 0 & 1-a-b \\ 0 & 0 & 1-c-d \\ 0 & 0 & 1-g-h \end{bmatrix}$$

$$Y = \begin{bmatrix} a & 0 & 0 \\ c & 0 & 0 \\ g & 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & b & 0 \\ 0 & d & 0 \\ 0 & h & 0 \end{bmatrix} \tag{1.6}$$

Note that $P_U = X + Y + Z$.

We shall exploit the structure of the submatrices defined in (1.6) and the transition probability matrix P to obtain explicit expressions for the distribution characteristic of CRW.

In section 2 we obtain the limiting distribution of the CRW with transition probability matrix (1.5) and related first passage characteristics. In Section 3 similar properties are investigated for the CRW when state space is made finite by placing restrictions on both sides.

2. CRW Restricted on One Side

The condition for the existence of an equilibrium solution is given by the following theorem.

Theorem 2.1

The necessary and sufficient condition for the Markov chain with he transition probability matrix (1.5) to have an equilibrium solution is given by

$$\frac{a(1-d)+bc}{(1-d)(1-g)-hc} < 1 \ . \tag{2.1}$$

Proof: Following Neuts (1981, p. 32), the necessary and sufficient condition for the existence of the equilibrium solution may be stated as

$$\underline{\pi}Y\underline{e} < \underline{\pi}X\underline{e} \tag{2.2}$$

where $\underline{\pi}$ is the limiting distribution of the Markov chain $\{U_n\}$ as given in (1.3). Going back to the equations $\underline{\pi}P_U = \underline{\pi}$, i.e., $\underline{\pi}(X+Y+Z) = \underline{\pi}$, we observe,

$$\begin{aligned} \underline{\pi}Y\underline{e} &= \pi_{+1} \\ \underline{\pi}X\underline{e} &= \pi_{-1} \end{aligned} \tag{2.3}$$

Thus, the condition (2.2) reduces to

$$\pi_{+1} < \pi_{-1} \tag{2.4}$$

Substituting from (1.3),

$$hc + (1-d)g < (1-a)(1-d) - bc$$

which on simplification reduces to the condition (2.1). □

Let

$$\underline{x} = (\underline{x}_0, \underline{x}_1, \underline{x}_2, \ldots) \tag{2.5}$$

where $\underline{x}_0$ is a two element row vector and $\underline{x}_i$ $(i = 1,2,\ldots)$ are three element row vectors, be the limiting distribution of the Markov chain. Following Neuts (1981, p. 25) we have

$$\underline{x}_i = \underline{x}_1 R^{i-1}, \quad i = 1, 2, \ldots \tag{2.6}$$

where R is the rate matrix of the Markov chain P. It is obtained in the following lemma.

<u>Lemma 2.1</u>

$$R = \begin{bmatrix} a & a\alpha & a\beta \\ c & c\alpha & c\beta \\ g & g\alpha & g\beta \end{bmatrix} \tag{2.7}$$

$$R^j = \beta^{j-1} R \qquad j = 1, 2, 3 \ldots \tag{2.8}$$

where

$$\alpha = \frac{ah+b(1-g)}{(1-d)(1-g)-hc}, \quad \beta = \frac{bc+a(1-d)}{(1-d)(1-g)-hc} \tag{2.9}$$

Proof: The rate matrix R satisfies the matrix equation

$$R = Y + RZ + R^2X$$

which can be written as

$$R = Y(I-Z)^{-1} + R^2X(I-Z)^{-1} \tag{2.10}$$

Let R_n be the nth iterate of R, such that

$$R_n = Y(I-Z)^{-1} + R_{n-1}^2X(I-Z)^{-1} \tag{2.11}$$

We have,

$$(I-Z)^{-1} = \begin{bmatrix} 1 & b/(1-d) & 0 \\ 0 & 1/(1-d) & 0 \\ 0 & h/(1-d) & 1 \end{bmatrix}$$

Substituting in (2.11) we get

$$R_n = R_{n-1}^2 \begin{bmatrix} 0 & \frac{h(1-a-b)}{1-d} & 1-a-b \\ 0 & \frac{h(1-c-d)}{1-d} & 1-c-d \\ 0 & \frac{h(1-g-h)}{1-d} & 1-g-h \end{bmatrix} + \begin{bmatrix} a & \frac{ab}{1-d} & 0 \\ c & \frac{cb}{1-d} & 0 \\ g & \frac{gb}{1-d} & 0 \end{bmatrix} \tag{2.12}$$

By induction we can show that, for $n \geq 1$,

$$R_n = \begin{bmatrix} a & \frac{a}{1-d}(b+hK_n) & aK_n \\ c & \frac{c}{1-d}(b+hK_n) & cK_n \\ g & \frac{g}{1-d}(b+hK_n) & gK_n \end{bmatrix} \tag{2.13}$$

where K_n satisfies the recurrence relation

$$K_n = \left[a + \frac{bc}{1-d} + (g + \frac{hc}{1-d}) K_{n-1} \right] \left[1-a - \frac{bc}{1-d} + \left(1-g - \frac{hc}{1-d}\right) K_{n-1} \right] \tag{2.14}$$

$$K_1 = 0$$

As $n \to \infty$, $R^n \to R$; hence $K_n \to K$ as well. Thus K can be determined by solving

$$K = \left[a + \frac{bc}{1-d} + \left(g + \frac{hc}{1-d}\right) K \right] \left[1-a - \frac{bc}{1-d} + \left(1-g - \frac{hc}{1-d}\right) K \right]$$

$$0 = \left(a + \frac{bc}{1-d}\right)\left(1-a - \frac{bc}{1-d}\right) + \left[\left(a + \frac{bc}{1-d}\right)\left(1-g - \frac{hc}{1-d}\right) - 1 \right.$$

$$\left. + \left(1-a - \frac{bc}{1-d}\right) \left(g + \frac{hc}{1-d}\right) \right] K + \left(g + \frac{hc}{1-d}\right)\left(1-g - \frac{hc}{1-d}\right)K^2 \tag{2.15}$$

After simplification the discriminant of equation (2.15) can be written as

$$= \left[\frac{c(b+h)}{1-d} + a + g-1 \right]^2 \tag{2.16}$$

and the two possible roots of the equation are

$$K = \frac{a(1-d)+bc}{(1-g)(1-d)-hc} \quad , \quad \frac{(1-a)(1-d)-bc}{g(1-d)+hc} \tag{2.17}$$

From (2.6) we may note that $R^j \to 0$ as $j \to \infty$. Using this condition we can show that the second solution is inadmissible. The lemma now follows using the first solution in (2.13) and simplifying. □

We may also note that the condition (2.1) is necessary for $R^j \to 0$ as $j \to \infty$.

Using R, the equilibrium solution is determined in the following theorem.

<u>Theorem 2.2</u>

When

$$\frac{a(1-d)+bc}{(1-d)(1-g)-hc} < 1$$

the limiting distribution of CRW is given by

$$\underline{x} = (\underline{x}_0, \underline{x}_1, \underline{x}_2, \ldots\ )$$

where

$$\underline{x}_0 = \frac{(1-d)(1-a-g) - c(h+b)}{((1-d)(1-g)-hc)(2(1-d)+h+b)} [h, (1-d)] \tag{2.18}$$

$$\underline{x}_j = \frac{(1-d)((1-d)(1-a-g) - c(h+b))}{((1-d)(1-g)-hc)(2(1-d)+h+b)} \left[\frac{a(1-d)+bc}{(1-d)(1-g)-hc}\right]^{j-1}$$

$$\left[1, \frac{b(1-g)+ah}{(1-d)(1-g)-hc} , \frac{a(1-d)+bc}{(1-d)(1-g)-hc}\right]$$

$$j = 1, 2, \ldots. \tag{2.19}$$

Proof: Writing out $\underline{x}P = \underline{x}$, we have

$$\begin{aligned} &\underline{x}_0 A + \underline{x}_1 C = \underline{x}_0 \\ &\underline{x}_0 B + \underline{x}_1 Z + \underline{x}_2 X = \underline{x}_1 \\ &\underline{x}_{i-1} Y + \underline{x}_i Z + \underline{x}_{i+1} X = \underline{x}_i \quad i \geq 2 \end{aligned} \tag{2.20}$$

From the second equation in (2.20) and equation (2.6) we get

$$\underline{x}_1 = \underline{x}_0 B\ (I - Z - RX)^{-1} \tag{2.21}$$

and

$$\underline{x}_j = \underline{x}_0 B\ (I - Z - RX)^{-1} R^{j-1}, \quad j \geq 1 \tag{2.22}$$

From the first equation in (2.20) and the normalizing condition $\underline{x}\ \underline{e} = 1$, we see that $\underline{x}_0$ is the solution of the following reduced system

$$\left.\begin{aligned} &\underline{x}_0\ (A+B\ (I - Z - RX)^{-1} C) = \underline{x}_0 \\ &\underline{x}_0 \underline{e} + \underline{x}_0 B\ (I - Z - RX)^{-1} (I-R)^{-1} \underline{e} = 1 \end{aligned}\right\} \tag{2.23}$$

where we have assumed that the unit column vectors $\underline{e}$ have appropriate dimensions.

Substituting from (1.6) and (2.7) for A,B,C,X,Z and R and simplifying we get the following results

$$(I - Z - RX)^{-1} = \begin{bmatrix} 1 & (b(1-g)+ah)\eta & (a(1-d)+bc)\eta \\ 0 & (1-g)\eta & c\eta \\ 0 & h\eta & (1-d)\eta \end{bmatrix} \tag{2.24}$$

where

$$\eta = ((1-d)(1-g)-hc)^{-1}.$$

$$A + B\,(I - Z - RX)^{-1}C = \begin{bmatrix} d & 1-d \\ h & 1-h \end{bmatrix} \tag{2.25}$$

$$(I - R)^{-1} = I + \frac{(1-d)(1-g)-hc}{(1-d)(1-a-g)-c(h+b)}\,R \tag{2.26}$$

$$B(I - Z - RX)^{-1}(I - R)^{-1} = \frac{(1-d)(1-g)-hc}{(1-d)(1-a-g)-c(h+b)}\,B(I - Z - RX)^{-1}$$

$$B(I - Z- RX)^{-1}(I - R)^{-1}\underline{e} = \frac{(1-d+b)(1-g+a)+(h-b)(a-c)}{(1-d)(1-a-g)-c(h+b)} \begin{bmatrix} 1-d \\ 1-h \end{bmatrix} \tag{2.27}$$

The results (2.18) and (2.19) now follows after substituting these results in (2.22) and (2.23) and simplifying. □

We close this section by giving a simple result on a first passage problem. Consider two complementary sets of states

$$S_i = \{(0,0),(0,-1),(1,+1),(1,0),(1,-1),\ldots,(i,+1),(i,0),(i,-1)\}$$

$$S_i^c = \{(i+1,+1),(i+1,0),(i+1,-1),\ldots\ldots\} .$$

it is well known that the elements of the matrix $B^*(I-D^*)^{-1}C^*$ give the first return probability (in the sense described above) into states y_1. If $\underline{x}_1^*$ and $\underline{x}_2^*$ are the limiting probabilites corresponding to sets of states y_1 and y_2, using reduced systems (see Lal and Bhat (1987, 1988)) we get the relation

$$\underline{x}_2^* = \underline{x}_1^* B^*(1-D^*)^{-1} . \tag{2.31}$$

Using the rate matrix R and equation (2.6), (2.31) can also be written as

$$\underline{x}_2^* = \underline{x}_1^* \begin{bmatrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \\ R & R^2 & \cdots \end{bmatrix} \tag{2.32}$$

Comparing (2.31) with (2.32) and noting that

$$C^* = \begin{bmatrix} 0 & 0 & \cdots & X \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2.33}$$

we get the result in the theorem. □

When $S_i = \{(0,0),(0,-1)\}$, for $B^*(I-D^*)^{-1}C^*$ we have $B(I-Z-RX)^{-1}C$, which simplifies to

$$\begin{bmatrix} 0 & 1-d \\ 0 & 1-h \end{bmatrix} \tag{2.34}$$

When we combine these probabilities with the elements of

$$A = \begin{bmatrix} d & 0 \\ h & 0 \end{bmatrix}$$

we find, as we should, that the recurrence probabilities to these states are 1. Further, using arguments similar to those used above we can show that $B(I-Z-RX)^{-1}R^{j-1}$ gives the expected number of visits to states $\{(j,+1), (j,0),(j,-1)\}$ before first passage to $\{(0,0),(0,-1)\}$ while avoiding it in the meantime, after having started initially from it. We have

$$B(I-Z-RX)^{-1}R^{j-1} = \beta^{j-1} \begin{bmatrix} 1-d & (1-d)\alpha & (1-d)\beta \\ 1-h & (1-h)\alpha & (1-h)\beta \end{bmatrix} \tag{2.35}$$

where α and β are given in (2.9). Thus, the mean recurrence time of the set $\{(0,0),(0,-1)\}$ is obtained as

$$= \frac{1}{1-\beta} \begin{bmatrix} (1-d)(1+\alpha+\beta) \\ (1-h)(1+\alpha+\beta) \end{bmatrix}$$

$$= \frac{(1-d+b)(1-g+a) + (h-b)(a-c)}{(1-d)(1-a-g) - c(h+b)} \begin{bmatrix} 1-d \\ 1-h \end{bmatrix} \tag{2.36}$$

3. CRW Restricted on Both Sides

Let the walk be restricted to the states $\{0,1,2,\ldots N\}$. Then, the transition probability matrix takes the form

$$
P = \begin{array}{c}
\\ (0,0) \\ (0,-1) \\ (1,+1) \\ (1,0) \\ (1,-1) \\ \vdots \\ (N-1,+1) \\ (N-1,0) \\ (N-1,-1) \\ (N,+1) \\ (N,0)
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccc}
(0,0) & (0,-1) & (1,+1) & (1,0) & (1,-1) & \cdots & (N-1,+1) & (N-1,0) & (N-1,-1) & (N,+1) & (N,0)
\end{array} \\
\left[\begin{array}{cc|ccc|c|ccc|cc}
d & 0 & 1-d & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
h & 0 & 1-h & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1-a-b & 0 & b & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 1-c-d & 0 & d & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 1-g-h & 0 & h & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & & & & & & & & \\
\hline
0 & 0 & 0 & 0 & 0 & \cdots & 0 & b & 0 & a & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & d & 0 & c & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & h & 0 & g & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1-b & 0 & b \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1-d & 0 & d
\end{array}\right]
\end{array}
$$

Using the submatrices defined in (1.6) and

$$
E = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}
$$

$$F = \begin{bmatrix} 0 & 0 & 1-b \\ 0 & 0 & 1-d \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} a & 0 \\ c & 0 \\ g & 0 \end{bmatrix}$$

we may represent P as

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-2 \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & \cdots & N-2 & N-1 & N \end{array} \\ \begin{bmatrix} A & B & 0 & 0 & \cdots & 0 & 0 & 0 \\ C & Z & Y & 0 & \cdots & 0 & 0 & 0 \\ 0 & X & Z & Y & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & Z & Y & 0 \\ 0 & 0 & 0 & 0 & \cdots & X & Z & G \\ 0 & 0 & 0 & 0 & \cdots & 0 & F & E \end{bmatrix} \end{array} \tag{3.1}$$

The limiting distribution $\underline{x} = (\underline{x}_0, \underline{x}_1, \underline{x}_2, \ldots \underline{x}_N)$ of the CRW can be determined using a backward recursion (see, Lal and Bhat, 1987, 1988) as in the following theorem. Note that

$$\underline{x}_0 = (x_{00}, x_{0,-1})$$

$$\underline{x}_i = (x_{i,+1}, x_{i,0}, x_{i,-1}) \qquad i = 1,2,\ldots,N-1$$

$$\underline{x}_N = (x_{N,+1}, x_{N,0})$$

Theorem 3.1

$$x_{00} = \left[1 + \frac{b}{h}\beta^{N-1} + \frac{1-d}{h}\left(2\frac{1-\beta^N}{1-\beta} + \beta\,\frac{1-\beta^{N-1}}{1-\beta}\right)\right]^{-1} \tag{3.2}$$

$$x_{i,+1} = \left(\frac{1-d}{h}\right)\beta^{i-1}x_{00} \qquad i = 1, 2, \ldots, N$$

$$x_{i,0} = \left(\frac{1-d}{h}\right)\alpha\beta^{i-1}x_{00} \qquad i = 1, 2, \ldots, N-1$$

$$x_{i,-1} = \left(\frac{1-d}{h}\right)\beta^{i}x_{00} \qquad i = 0, 1, 2, \ldots, N-1$$

$$x_{N,0} = \left(\frac{b}{h}\right)\beta^{N-1}x_{00} \tag{3.3}$$

where

$$\alpha = \frac{ah+b(1-g)}{(1-d)(1-g)-hc} \quad , \quad \beta = \frac{bc+a(1-d)}{(1-d)(1-g)-hc} \tag{3.4}$$

as defined in (2.9).

Proof: Clearly we have

$$\underline{x}\ P = \underline{x} \tag{3.5}$$

$$\underline{x}\ \underline{e} = 1 \tag{3.6}$$

To use the backward recursion, we start with the last 4 equations in (3.5).

$$x_{N,0} = bx_{N,+1} + dx_{N,0}$$

$$x_{N,+1} = ax_{N+1,+1} + cx_{N-1,0} + gx_{N-1,-1}$$

$$x_{N-1,-1} = (1-b)x_{N,+1} + (1-d)x_{N,0}$$

$$x_{N-1,0} = bx_{N-1,+1} + dx_{N-1,0} + hx_{N-1,-1}$$

From these equations, the last 4 variables can be expressed in terms of $x_{N-1,+1}$ as

$$\begin{aligned} x_{N,0} &= \frac{b}{1-d}\,\beta\, x_{N-1,+1} \\ x_{N,+1} &= \beta\, x_{N-1,+1} \\ x_{N-1,-1} &= \beta\, x_{N-1,+1} \\ x_{N-1,0} &= \alpha\, x_{N-1,+1} \end{aligned} \tag{3.7}$$

where α and β are defined in (3.4). Next use three equations preceding the last 4 in (3.5).

$$x_{N-1,+1} = ax_{N-2,+1} + cx_{N-2,0} + gx_{N-2,-1}$$

$$x_{N-2,-1} = (1-a-b)x_{N-1,+1} + (1-c-d)x_{N-1,0} + (1-g-h)x_{N-1,-1}$$

$$x_{N-2,0} = bx_{N-2,+1} + dx_{N-2,0} + hx_{N-2,-1}$$

Now expressing all the variables in terms $x_{N-2,+1}$ and using (3.7), from these equations we get

$$
\begin{aligned}
x_{N0} &= \left(\frac{b}{1-d}\right)\beta^2\, x_{N-2,+1} \\
x_{N,+1} &= \beta^2\, x_{N-2,+1} \\
x_{N-1,-1} &= \beta^2\, x_{N-2,+1} \\
x_{N-1,0} &= \alpha\beta\, x_{N-2,+1} \\
x_{N-1,+1} &= \beta\, x_{N-2,+1} \\
x_{N-2,-1} &= \beta\, x_{N-2,+1} \\
x_{N-2,0} &= \alpha\, x_{N-2,+1}
\end{aligned}
\tag{3.8}
$$

On repeatedly using the above recursive procedure we get

$$
\begin{aligned}
x_{N,0} &= \left(\frac{b}{1-d}\right)\beta^{N-1} x_{1,+1} \\
x_{i,0} &= \alpha\beta^{i-1}\, x_{1,+1} && i = 1, 2, \ldots, N-1 \\
x_{i,+1} &= \beta^{i-1}\, x_{1,+1} && i = 1, 2, \ldots, N \\
x_{i,-1} &= \beta^{i}\, x_{1,+1} && i = 0, 1, 2, \ldots, N-1
\end{aligned}
\tag{3.9}
$$

$$x_{1,+1} = \left(\frac{1-d}{h}\right) x_{00}$$

The theorem now follows when we use (3.9) in the normalizing condition (3.6). □

In random walk problems first passage problems are also of interest. For instance, in the context of a gambler's ruin problem, the probability of ruin, the probability of win and the expected duration of the game are all related to the first passage of the walk to a particular state. In the case of CRW with stay similar problems can also be investigated. A computational procedure to obtain these characteristics is to determine what has come to be known as the fundamental matrix of the appropriate transition probability matrix. See Kemeny and Snell (1960) for the definition of the matrix and the interpretation of results. The basic fundamental matrix for the finite state CRW can be given as

$$M = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & \cdots & N-2 & N-1 & N \end{matrix} \\ \begin{bmatrix} I-Z & -Y & 0 & \cdots & 0 & 0 & 0 \\ -X & I-Z & -Y & \cdots & 0 & 0 & 0 \\ 0 & -X & I-Z & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & & -X & I-Z & -Y \\ 0 & 0 & 0 & & 0 & -X & I-Z \end{bmatrix} \end{array}^{-1} \tag{3.10}$$

Let M_{ij} be the (i,j)th element of M. We have

Lemma 3.1

$$M_{ij} = \sum_{\ell=0}^{\min(i-1,j-1)} \left(\prod_{q=\ell+1}^{i-1} X_{N-q} \right) R_{N-\ell} \left(\prod_{s=\ell+1}^{j-1} Y_{N-s} \right) \tag{3.11}$$

where R_ℓ satisfies the relation

$$\begin{aligned} R_\ell &= (I - Z - YR_{\ell-1}X)^{-1} \qquad \ell = 2,3,\ldots, N \\ R_1 &= (I - Z)^{-1}, \quad R_0 = 0 \end{aligned} \tag{3.12}$$

and

$$Y_\ell = YR_\ell \quad \text{and} \quad X_\ell = R_\ell X, \qquad \ell = 1, 2, \ldots, N. \tag{3.13}$$

In (3.11) $\prod_{q=\ell+1}^{i-1} X_{N-q}$ is taken in pre-order as q increases, while $\prod_{s=\ell+1}^{j-1} Y_{N-s}$ is taken in post-order as s increases.

Proof: The result follows from the recursive algorithm following Theorem 2.2 of Lal and Bhat (1988). ∞

Theorem 3.2

$$M_{ij} = \begin{bmatrix} \mu_{ij}^{(11)} & \mu_{ij}^{(12)} & \mu_{ij}^{(13)} \\ \mu_{ij}^{(21)} & \mu_{ij}^{(22)} & \mu_{ij}^{(23)} \\ \mu_{ij}^{(31)} & \mu_{ij}^{(32)} & \mu_{ij}^{(33)} \end{bmatrix} \tag{3.14}$$

where

$$\mu_{ij}^{(11)} = S_{ij}^{0} + \gamma S_{ij}^{5} + \nu S_{ij}^{7}$$

$$\mu_{ij}^{(12)} = S_{ij}^{4} + hK_{N-i+1}S_{ij}^{6} + \left[\frac{b+(ah-bg)K_{N-j}}{1-d-\nu K_{N-j}}\right]\left(\gamma S_{ij}^{5} + \nu S_{ij}^{7}\right)$$

$$\mu_{ij}^{(13)} = \gamma S_{ij}^{2} + (1-d)K_{N-i+1}S_{ij}^{6} + \left[\frac{\gamma K_{N-j}}{1-d-\nu K_{N-j}}\right]\left(\gamma S_{ij}^{5} + \nu S_{ij}^{7}\right)$$

$$\mu_{ij}^{(21)} = cS_{ij}^{5} + \gamma S_{ij}^{7}\left[1 + \frac{c(K_{N-i}-1)}{1-d-\nu K_{N-i}}\right]$$

$$\mu_{ij}^{(22)} = S_{ij}^{3} + hS_{ij}^{6}\left[1 + \frac{c(K_{N-i}-1)}{1-d-\nu K_{N-i}}\right] +$$

$$\left[\frac{b+(ah-bg)K_{N-j}}{1-d-\nu K_{N-j}}\right]\left[cS_{ij}^{5} + \nu S_{ij}^{7}\left(1 + \frac{c(K_{N-i}-1)}{1-d-\nu K_{N-i}}\right)\right]$$

$$\mu_{ij}^{(23)} = cS_{ij}^{2} + (1-d)S_{ij}^{6}\left[1 + \frac{c(K_{N-i}-1)}{1-d-\nu K_{N-i}}\right] +$$

$$\left[\frac{\gamma K_{N-j}}{1-d-\nu K_{N-j}}\right]\left[cS_{ij}^{5} + \nu S_{ij}^{7}\left(1 + \frac{c(K_{N-i}-1)}{1-d-\nu K_{N-i}}\right)\right]$$

$$\mu_{ij}^{(31)} = \nu\left[S_{ij}^{5} + S_{ij}^{7}\left(\frac{1-d-\nu}{1-d-\nu K_{N-i}}\right)\right]$$

$$\mu_{ij}^{(32)} = h\left[S_{ij}^{1} + S_{ij}^{6}\left(\frac{1-d-\nu}{1-d-\nu K_{N-i}}\right)\right] +$$

$$\nu \left[\frac{b+(ah-bg)K_{N-j}}{1-d-\nu K_{N-j}} \right] \left[S^5_{ij} + S^7_{ij} \left(\frac{1-d-\nu}{1-d-\nu K_{N-i}} \right) \right]$$

$$\mu^{(33)}_{ij} = (1-d) \left[S^1_{ij} + S^6_{ij} \left(\frac{1-d-\nu}{1-d-\nu K_{N-i}} \right) \right] +$$

$$\nu \left[\frac{\nu K_{N-j}}{1-d-\nu K_{N-j}} \right] \left[S^5_{ij} + S^7_{ij} \left(\frac{1-d-\nu}{1-d-\nu K_{N-i}} \right) \right]$$

with

$$\begin{aligned} \gamma &= bc + a(1-d) \\ \nu &= hc + g(1-d) \quad , \end{aligned} \tag{3.15}$$

K_ℓ defined by the recursion

$$\begin{aligned} K_\ell &= 1 + \frac{\gamma(K_{\ell-1}-1)}{1-d-\nu K_{\ell-1}} \qquad \ell = 1, 2, \ldots \\ K_0 &= 0 \quad , \end{aligned} \tag{3.16}$$

and

$$S^0_{ij} = \sum_0^{IJ} \delta^{**}_{i,\ell+2} \delta^{**}_{j,\ell+2}$$

$$S^1_{ij} = \sum_0^{IJ} \delta^{**}_{i,\ell+2} \delta^{**}_{j,\ell+2} (1-d-\nu K_{N-\ell-1})^{-1}$$

$$S^2_{ij} = \sum_0^{IJ} \delta^{**}_{i,\ell+2} \delta^{**}_{j,\ell+2} (1-d-\nu K_{N-\ell-1})^{-1} K_{N-\ell-1}$$

$$S^3_{ij} = \sum_0^{IJ} \delta^{**}_{i,\ell+2} \delta^{**}_{j,\ell+2} (1-d-\nu K_{N-\ell-1})^{-1} (1-gK_{N-\ell-1})$$

$$S^4_{ij} = \sum_0^{IJ} \delta^{**}_{i,\ell+2} \delta^{**}_{j,\ell+2} (1-d-\nu K_{N-\ell-1})^{-1} (b+(ah-bg)K_{N-\ell-1})$$

The theorem now follows by substituting from (3.20) - (3.23) in (3.11) and simplifying. □

The fundamental matrix derived in Theorem 3.2 leads directly to the following first passage characteristics: (i) first passage probabilities to states {(0,0), (0,-1)} and {(N,+1),(N,0)} while avoiding each other, (ii) mean and variance of first passage times of such transitions, and (iii) the first passage probabilities into states identified in (i), but not necessarily avoiding each other. To determine the first two characteristics we convert states 0 and N of (3.1) into absorbing states, and re-arrange and re-partition the matrix P as

$$P = \begin{array}{c} \\ 0 \\ N \\ 1 \\ 2 \\ \vdots \\ N-2 \\ N-1 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & N & 1 & 2 & 3 & \cdots & N-1 \end{array} \\ \left[\begin{array}{cc|ccccc} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \hline C & 0 & Z & Y & 0 & \cdots & 0 \\ 0 & 0 & X & Z & Y & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & Y \\ 0 & G & 0 & 0 & 0 & \cdots & Z \end{array}\right] \end{array} \tag{3.24}$$

$$= \left[\begin{array}{c|c} I & 0 \\ \hline L & H \end{array}\right]$$

The corresponding fundamental matrix is

$$M = (I-H)^{-1}$$

as derived in Theorem 3.2, with N replaced by N-1. Its row sums give the expected time the process takes to enter 0 or N, having started at the corresponding state. The fundamental matrix can also be used to compute variances of first passage times (see Kemeny and Snell, 1960). The first passage probabilities to states 0 and N from any one of the intitial states are obtained as the elements of the matrix product ML.

Finally, to determine first passage characteristics of the process to state 0 (or N), allowing the process to visit state N (or 0) in the mean time, we have to extend the matrix H of (3.24) as follows.

$$H^{(1)} = \left[\begin{array}{cccccc|cc} & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & \vdots & \vdots \\ & & & H & & & 0 & 0 \\ & & & & & & a & 0 \\ & & & & & & c & 0 \\ & & & & & & g & 0 \\ \hline 0 & 0 & \cdots & 0 & & 1-b & 0 & b \\ 0 & 0 & \cdots & 0 & & 1-d & 0 & d \end{array}\right]$$

or

$$H^{(2)} = \left[\begin{array}{cc|ccccc} d & 0 & 1-d & 0 & 0 & \cdots & 0 \\ h & 0 & 1-h & 0 & 0 & \cdots & 0 \\ \hline 0 & 1-a-b & & & & & \\ 0 & 1-c-d & & & & & \\ 0 & 1-g-h & & & & & \\ 0 & 0 & & & H & & \\ \vdots & \vdots & & & & & \\ 0 & 0 & & & & & \\ 0 & 0 & & & & & \end{array}\right]$$

The fundamental matrix $M^{(1)} = (I-H^{(1)})^{-1}$ is used while considering the first passage to state 0 and $M^{(2)} = (I-H^{(2)})^{-1}$ is used while considering the first passage to state N. In these cases the results of Theorem 3.2 can be extended using methods of Theorem 2.2 of Lal and Bhat (1988), and the elements $M_{ij}^{(1)}$ and $M_{ij}^{(2)}$ of $M^{(1)}$ and $M^{(2)}$ can be expressed in terms of elements M_{ij} of equations (3.14) - (3.17). We shall not give them here due to their cumbersome nature.

The first passage results given by M_{ij} of (3.14) - (3.17) may look involved and complicated. Nevertheless, as anyone well-versed in scientific computations can realize they can be conveniently used to get useful numerical results.