CONTROL CHARTS FOR MARKOV DEPENDENT PRODUCTION PROCESSES

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ABSTRACT

Upper and lower control limits for control charts for Markov dependent production processes are obtained using the distribution of the number of successes in Markov trials derived by the authors. The standard k-sigma limits as well as control limits based on the exact distribution are obtained. Algorithms for the determination of sample size are given with illustrations.

Key words: Control charts, quality control, Markov dependent production process, Markov chains.

1. Introduction

A control chart is an important tool in monitoring production processes in order to control product quality. Any book on quality control gives procedures to set up and use attribute control charts when items are classified as defective (or nonconforming) or non-defective (or conforming). See Duncan (1974), Grant and Leavenworth (1980), Wadsworth et al. (1986), to cite only a few of such books. In all such treatments as well as those we may find in all research papers on the subject, the production process considered is an independent process in which the quality of one item is not dependent on the quality of the preceding one. However, often in a production process, the quality of items is serially dependent (see for instance, Broadbent (1958)). With this in mind we propose here procedures to determine the upper and lower control limits of a control chart for a production process which is Markov dependent.

In a recent paper (Bhat and Lal, 1987a) the authors have derived distribution characteristics of the number of successes in a sequence of Markov trials. This is done by defining an augmented Markov chain whose state space includes the information on the number of successes. We describe this procedure in section 2. In section 3, the control limits are obtained and in section 4, its operating characteristic curve is discussed. Finally the sample size problem is addressed in the last section. We may note here that in a companion paper (Bhat and Lal, 1987b), the authors have proposed a sequential inspection plan for monitoring Markov dependent production processes, which is a modification of the standard acceptance sampling procedure.

2. An Augmented Markov Chain

Let $\{Y_n,\ n=0,1,2,\ldots\}$ be a two state Markov chain with states 0 and 1. We assume that the quality attributes of the nth inspected item can be represented by Y_n, with states 0 and 1 representing good (acceptable) and bad (unacceptable) respectively. Let the transition probability p_{ij} be defined as

$$p_{ij} = P[Y_{n+1}=j \mid Y_n = i] \qquad n = 0,1,2,\ldots. \qquad i,j = 0,1.$$

It is well known that the process is completely specified by the transition probability matrix

$$P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \qquad (1 \geq a,\ b \geq 0) \tag{1}$$

along with the initial state Y_0.

Let

$$p_{ij}^{(n)} = P(Y_n = j \mid Y_0 = i). \qquad n = 1,2,\ldots \tag{2}$$

In a two state Markov chain defined as in (1), when $|1 - a - b| < 1$ the limiting probabilities

$$\pi_j = \lim_{n\to\infty} p_{ij}^{(n)} \quad , \quad (j = 0,1)$$

exist independent of the initial state and are given by

$$(\pi_0,\pi_1) = \left(\frac{b}{a+b}\ ,\ \frac{a}{a+b}\right)\ . \tag{3}$$

Also if we denote by ρ_j the serial correlation (of lag j) of the process for which the initial state Y_0 has the distribution (3), we get

$$\rho_j = \rho^j, \text{ where } \rho = 1 - (a+b) \quad . \tag{4}$$

Corresponding to the fraction defective in a production process with sequential items independent of each other, in a Markov dependent process we may identify π_1, the probability that an item is defective in the long run. Thus a Markov dependent production process can be identified by two parameters, the fraction defective p $(= \pi_1)$ and the serial correlation ρ, with the following admissible ranges:

$$1 - \min\{\frac{1}{p}, \frac{1}{1-p}\} < \rho < 1$$

which can be stated also as

$$\max\{0, \frac{-\rho}{1-\rho}\} < p < \min\{1, \frac{1}{1-\rho}\} \quad . \tag{5}$$

Using p and ρ the transition probability matrix P can be represented as

$$P = \begin{bmatrix} 1 - p(1-\rho) & p(1-\rho) \\ (1-p)(1-\rho) & p+\rho(1-p) \end{bmatrix} \quad . \tag{6}$$

In order to avoid making decisions based on transient characteristics of the process, a sample of size n items is inspected after the process has attained stability. Accordingly we assume the initial state distribution to be $[(1-p),p]$. We may consider charts under two sampling schemes. In the first scheme, a series of samples of fixed size n are inspected with enough spacing between samples. Appropriate spacing j between samples can be determined so that lag j serial correlation ρ^j becomes negligible. This sampling scheme is similar to the Shewhart scheme usually adopted for the independent production process. In the second scheme, cumulative numbers of defectives is noted for every k additional items, starting with an initial sample of size n. In practice inspection can be stopped with a predesignated number of incremental samples. This is similar to the sample scheme adopted for CUSUM charts. In either scheme, to determine the control limits, we need the distribution of the number of successes in a specified number of Markov dependent trials. Since the distribution of the number of successes in dependent trials given by Gabriel (1959) using combinatorial arguments is not very convenient, in our calculations we use the results derived by the authors recently (Bhat and Lal, 1987a). These results are based on the following augmented Markov model.

Define a Markov chain (X_n, Y_n) where

X_n = number of defectives in n inspected items

Y_n = state at the nth inspection.

The state space of the Markov chain can be given as

$$\{00, 11, 10, 21, 20, 31, 30, \ldots\ldots\}$$

with the transition probabilities

$$p_{ij,k\ell} = P(X_n = k, Y_n = \ell \mid X_0 = i, Y_0 = j) \tag{7}$$

and the matrix $[p_{ij,k\ell}]$

$$
P = \begin{array}{c} \\ 00 \\ 11 \\ 10 \\ 21 \\ 20 \\ 31 \\ \cdot \\ \cdot \\ \cdot \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 00 & 11 & 10 & 21 & 20 & 31 & \cdot & \cdot & \cdot \end{array} \\
\left[\begin{array}{ccccccccc}
1-a & a & & & & & & & \\
& & b & 1-b & & & & & \\
& & 1-a & a & & & & & \\
& & & & b & 1-b & & & \\
& & & & 1-a & a & & & \\
& & & & & & \cdot & & \\
& & & & & & & \cdot & \\
& & & & & & & & \cdot \\
& & & & & & & &
\end{array}\right]
\end{array}
\qquad (8)
$$

The distribution of the number of defectives in a sample of size n is given by the elements of P^n, the n^{th} power of matrix P. Because of the triangular structure of P, P^n can be explicitly determined (Bhat and Lal, 1987a). It may be noted that for small values of n the distribution of the number of defectives can be easily obtained by simple matrix multiplication.

3. Control Limits

Let p be the specified standard for the fraction of defectives in a Markov dependent production process with a serial correlation (of lag one) equal to ρ. Using the transition probability structure given in equations (1) and (6) we shall write

$$p(1-\rho) = a \quad \text{and} \quad (1-p)(1-\rho) = b. \qquad (9)$$

It should be noted that the admissible ranges for ρ and p are defined in equation (5).

For a fraction defective chart p will be the central line and the upper and lower control limits can be determined, based on the distribution of the number of defectives in a sample size n, as follows. Let X_n be the number of defectives in a sample of size n, including the initial observation. Then

$$
\begin{aligned}
P(X_n = h) \\
&= (1-p) \sum_{j=\min(1,h)}^{h} \genfrac{}{}{0pt}{}{h-1}{j-1} (1-a)^{n-h-j-1}(1-b)^{h-j} a^j b^{j-1} \\
&\cdot \left(b\delta^{**}_{k+j-1,n-1}\binom{n-h-1}{j} + (1-a)\delta^{**}_{0h}\,\delta^{**}_{h+j-2,n-1}\binom{n-h-1}{j-1} \right. \\
&\left. + p\delta^{**}_{0h}(1-b)^{h-1}\,\delta_{h-1,n-1} + \delta^{**}_{0h,n-1}\,b(1-a)^{n-h-1} \right) \\
&+ p \sum_{j=1}^{h-1} \binom{h-1}{j} (1-a)^{n-h-j-1}(1-b)^{h-j-1} a^j b^j \\
&\cdot \left(b\delta^{**}_{h+j-1,n-1}\binom{n-h-1}{j} + (1-a)\delta^{**}_{h+j-2,n-1}\binom{n-h-1}{j-1} \right)
\end{aligned}
$$

$$h = 0,\ 1,\ 2,\ \ldots,\ n. \qquad (10)$$

where

$\delta_{0h} = 1$ if $h = 0$, and $=0$ otherwise

$\delta^{**}_{gh} = 1$ if $g < h$, and $=0$ if $g \geq h$

$\delta^{**}_{0gh} = \delta^{**}_{0g}\ \delta^{**}_{0h}\ (1 - \delta^{**}_{gh})$.

When the initial state distribution is given by (3), the mean and variance of X_n can be obtained as

$$E(X_n) = np \tag{11}$$

$$Var(X_n) = np(1-p) + 2p(1-p)\left(\frac{\rho}{1-\rho} \quad n - \frac{1-\rho^n}{1-\rho}\right) \tag{12}$$

From the distribution of X_n both kσ (k times the standard deviation) and 100(1-α)% limits (with 100(α/2)% of the distribution at either tail) can be numerically obtained by using the following procedure.

kσ Limits

$$(p - k\sqrt{Var(X_n)}/n, \quad p + k\sqrt{Var(X_n)}/n)$$

where $Var(X_n)$ is the variance of defectives in a sample of size n for given p and ρ.

100(1-α)% Limits

$$(IL/n, \ IU/n)$$

where $\sum_{h=0}^{h=IL} P(X_n = h) \geq \alpha/2$

and

$$\sum_{h=0}^{h=IU} P(X_n = h) \geq 1 - \alpha/2$$

Table 1
Upper control limits for Markovian Production Processes
(Lower control limits are 0 in all cases).

p	ρ	n	3σ limits		99% limits	
			p-chart	np-chart	p-chart	np-chart
.01	0.5	100	.06136	7	.09000	9
		125	.05601	7	.08000	10
		150	.05204	8	.06667	10
		175	.04895	9	.06286	11
		200	.04647	10	.06000	12
	0.0	100	.03986	4	.04000	4
		125	.03672	5	.04000	5
		150	.03439	6	.03333	5
		175	.03258	6	.03429*	6
		200	.03114	7	.03000	6
	-0.01	100	.03958	4	.04000	4
		125	.03645	5	.04000	5
		150	.03416	6	.03333	5
		175	.03238	6	.03429*	6
		200	.03093	7	.03000	6
.02	0.5	100	.09227	10	.12000	12
		125	.08473	11	.10400	13
		150	.07915	12	.09333	14
		175	.07480	14	.08571	15
		200	.07129	15	.08000	16
	0.0	100	.06202	7	.06000	6
		125	.05758	8	.05600	7
		150	.05431	9	.05333	8
		175	.05178	10	.05143	9
		200	.04972	10	.05000	10
	-0.02	100	.06119	7	.06000	6
		125	.05685	8	.05600	7
		150	.05364	9	.05333	8
		175	.05114	9	.05143	9
		200	.04916	10	.05000	10

* These limits do not follow the pattern because np-chart limits are determined first.

The procedures for the determination of control limits for the Shewhart chart and the CUSUM chart are similar. Because of the correlated nature of the process, the variation in the control limits

is nonlinear in n (or $\frac{1}{n}$) and therefore, the control limits for the

CUSUM chart can be obtained the same way as Table 1 in which cumulative number of defectives (or successive fraction defectives) for sample sizes increasing incrementally by a specified number are displayed. Nevertheless as a good approximation, a V-mask can also be constructed based on the following observation.

$$\frac{1}{n}\sqrt{\mathrm{Var}(X_n)} = \left[\frac{p(1-p)}{n} + \frac{2p(1-p)\rho}{n(1-\rho)} - \frac{2p(1-p)\rho(1-\rho^n)}{n^2(1-\rho)^2}\right]^{\frac{1}{2}} \tag{13}$$

For large n the last term in this expression can be considered negligible. After simplification, we then get

$$\frac{1}{n}\sqrt{\mathrm{Var}(X_n)} \cong \left[\frac{p(1-p)(1+\rho)}{n(1-\rho)}\right]^{\frac{1}{2}} \tag{14}$$

which is similar to the expression one gets in the independent process case except for the terms incorporating correlation.

4. Operating Characterstic Curve

The operating characteristic curve gives the probabilities of the process being under control for varying quality levels for a specified set of upper and lower control limits. If c_1 and c_2 are the lower and upper control limits designed for a quality level p_0, correlation ρ, and sample size n, the O.C. curve is obtained by graphing the probability (for different values of p)

$$P_a(p,\rho) = P(X_n \leq c_2) - P(X_n < c_1). \tag{15}$$

5. Sample Size

The methods of determining sample size for control charts are available in the literature. (We shall not include cost function techniques in our discussion). (i) Determine the sample size n such that the probability of obtaining no defectives in the sample has a prespecified value. (ii) Determine n such that the margin of error in estimating the quality level has a pre-specified value with a given confidence level. Under assumptions of normality, the method is also described as the one which there is a 50% chance of detecting a shift equal to the distance between the central line to the control limit. These two procedures can be employed in the dependent process case as follows.

(i) Let γ_0 be the prespecified probability of obtaining no defectives in a sample of size n. Using Markovian transition probabilities given by (6), we need n such that

$$\gamma_0 = (1-p)(1-p(1-\rho))^{n-1} \tag{18}$$

Solving for n,

$$n = 1 + [\ln(\gamma_0/(1-p))/ \ln(1-p(1-\rho))]. \tag{19}$$

Table 2 below presents sample sizes as determined by this approach in a few cases.

Table 2

Sample size when probability γ_0 of zero defectives is known.

p	ρ	γ_0 .01	γ_0 .05	γ_0 .10
.01	0.5	918	597	459
	0.0	459	299	230
	-0.01	454	296	227
.02	0.5	458	298	229
	0.0	228	149	114
	-0.02	224	146	112
.05	0.5	181	118	90
	0.0	90	59	45
	-0.05	86	56	43
.10	0.5	89	58	44
	0.0	44	29	22
	-0.10	40	26	20

(ii) Let d be the margin of error in estimate (or shift in the quality level). The sample size n can be determined based on two types of limits discussed earlier by solving the following equations.

a. kσ limits:

$$k\left(\frac{\sqrt{Var}(X_n)}{n}\right) = d \qquad (20)$$

where X_n is the number of defectives, whose distribution is given by (10).

b. $100(1-\alpha)\%$ limits:

$$(IU_n - IL_n)/n = d^* \tag{21}$$

where $d^* = \max(0,p-d) + \min(1,p+d)$

and $\sum_{h=IL_n}^{h=IU_n} P(X_n = h) = 1 - \alpha$.

When the sample size n is expected to be large, it can be approximated as

$$n = \frac{k^2 p(1-p)(1+\rho)}{d^2(1-\rho)} \tag{22}$$

in the case of (20), and

$$n = z^2_{1-\frac{\alpha}{2}} \frac{p(1-p)(1+\rho)}{d^2(1-\rho)} . \tag{23}$$

As one may expect, from Table 3 it can be seen that these approximations are good in the case of $k\sigma$ limits, but not so in the case of $100(1-\alpha)\%$ limits.

The procedures devised to solve for n in equations (20) and (21) use modified regula falsi method (see Conte and deBoor (1972)) to perform single variable fixed point iterations. Equations (20) (or (21)) is solved until it is satisfied within a certain prespecified error tolerance (ε) or n is the same for a certain prespecified number of iterations (ITER). Each implementation of modified regula falsi method requires determination of bounds for an iterative solution value of n. It assumes that n is real and the function in (20) (or (21)) is continuous in n, even though it is evaluated only for integer values of n. One should note that the function in (20) (or (21)) is monotonic in n which facilitates faster convergence.

The steps required to find n using equation (20) are given below

Step 1[Stop Parameter] $\varepsilon = .00001$, ITER = 5.

Step 2[Initial Guess] $n = \dfrac{k^2 p(1-p)(1+\rho)}{d^2(1-\rho)}$

Step 3[Bounds for n]

3a. If $\left| k\sqrt{Var(X_n)}/n - d \right| \leq \varepsilon$ then Step 5.

3b. If $k\sqrt{Var\ (X_n)}/n < d$ then go to 3f.

3c. $n_L = n$, $n_U = 3n/2$, $n = n_U$.

3d. If $\left| k\sqrt{Var(X_n)}/n - d \right| \leq \varepsilon$ then Step 5.

3e. If $k\sqrt{Var(X_n)}/n > d$ then go to 3c else go to Step 4.

3f. $n_U = n$, $n_L = 2n/3$, $n = n_L$.

3g. If $\left| k\sqrt{Var(X_n)}/n - d \right| \leq \varepsilon$ then Step 5.

3h. If $k\sqrt{Var(X_n)}/n < d$ then go to 3f.

Step 4[Solve for n] Solve $k\sqrt{Var(X_n)}/n = d$ for n using modified regula falsi iterative method with stopping rule being that the equation is satisfied within error tolerance ε or n remains the same for ITER number of iterations.

Step 5[Stop] Stop.

The above procedure to find n can be used for equation (21) by obtaining the initial guess for n as $n = z^2_{1-\frac{\alpha}{2}} \dfrac{p(1-p)(1+\rho)}{d^2(1-\rho)}$, and replacing $k\sqrt{Var(X_n)}$ by $(IU_n - IL_n)$. Table 3 presents sample size as determined by these approaches for a few cases.

Table 3

Sample size when margin of error d is specified.

(ε = .00001)

p	ρ	d	kσ limits				100(1-a)% limits			
			2σ		3σ		95%		99%	
			Exact	Appr.	Exact	Appr.	Exact	Appr.	Exact	Appr.
.05	0.5	.05	226	228	511	513	210	219	360	379
		.08	87	90	199	201	123	86	208	148
	0.0	.05	76	76	171	171	70	73	110	127
		.08	30	30	67	67	38	29	69	50
	-0.05	.05	68	69	154	155	68	67	110	115
		.08	26	27	60	61	30	26	61	45

REFERENCES

Bhat, U. Narayan and Lal, Ram (1987a), "Number of Successes in Markov Trials" SMU/DS/TR-211, Department of Statistical Science, SMU, Dallas, Texas 75275. Submitted for publication.

Bhat, U. Narayan and Lal, Ram (1987b), "A Sequential Inspection Plan for Markov Dependent Production Processes", SMU/DS/TR - 209 (Revised), Department of Statistical Science, SMU, Dallas, Texas 75275, Submitted for publication.

Broadbent, S.R. (1958), "The Inspection of a Markov Process", J. Roy. Statist. Soc. B20, 111-119.

Conte, S.D. and deBoor, C. (1972), Elementary Numerical Analysis, McGraw Hill, New York.

Duncan, A.J. (1974), Quality Control and Industrial Statistics, 4th ed., Richard D. Irwin, Inc., Homewood, IL

Gabriel, K.R. (1959), "The Distribution of the Number of Successes in a Sequence of Dependent Trials", Biometrika, 46, 454-460.

Grant, L. and Leavenworth, R.S. (1980), Statistical Quality Control, 5th Ed. , McGraw Hill, New York.

Wadsworth, H.M., Stephens, K.S., and Godfrey, A.B. (1986), Modern Methods for Quality Control and Improvement, John Wiley and Sons, Inc., New York.

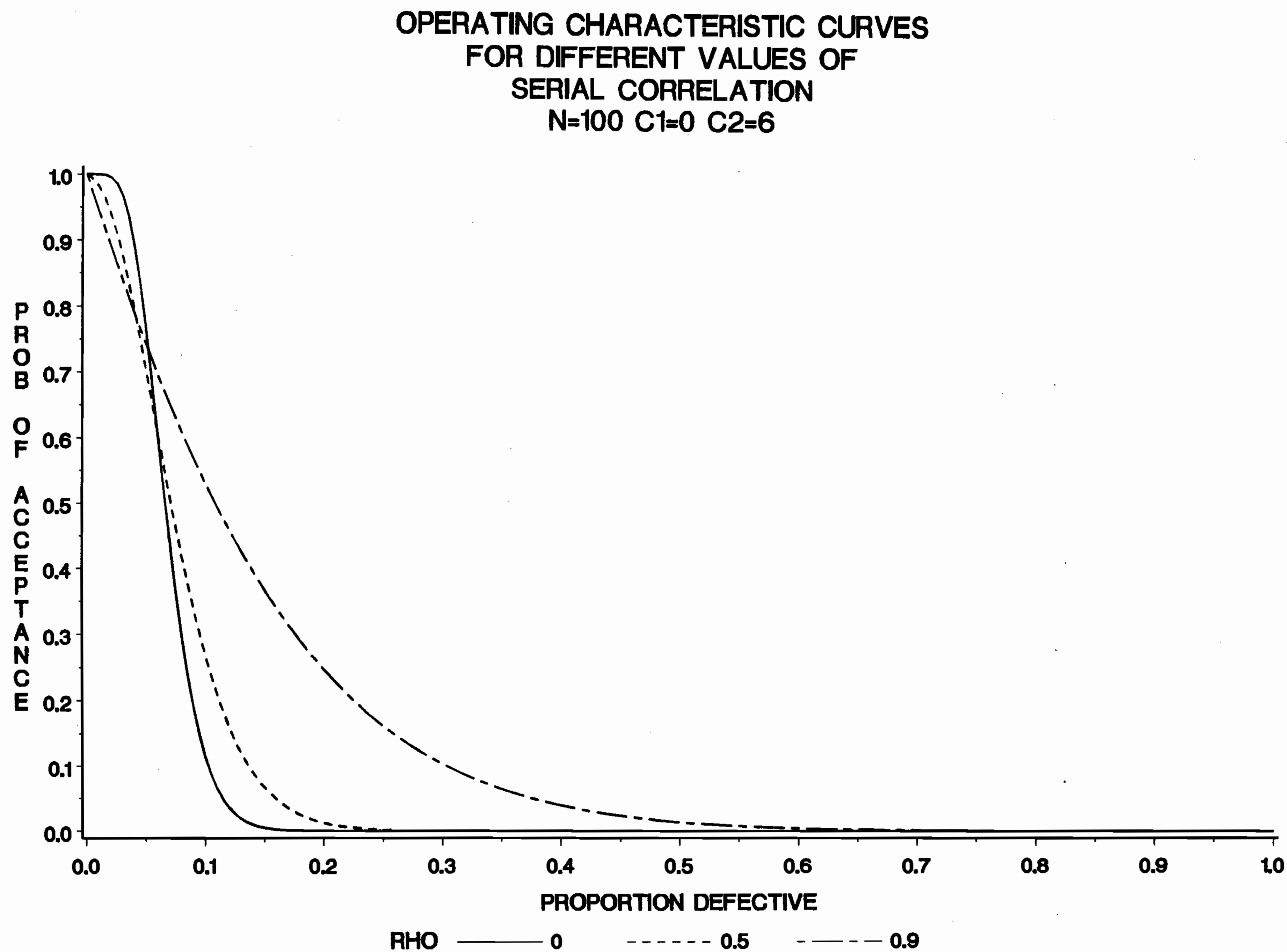
OPERATING CHARACTERISTIC CURVES
FOR DIFFERENT VALUES OF
SERIAL CORRELATION
N=100 C1=0 C2=6
PROB OF ACCEPTANCE
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
PROPORTION DEFECTIVE
RHO
0
0.5
0.9

Figure 1

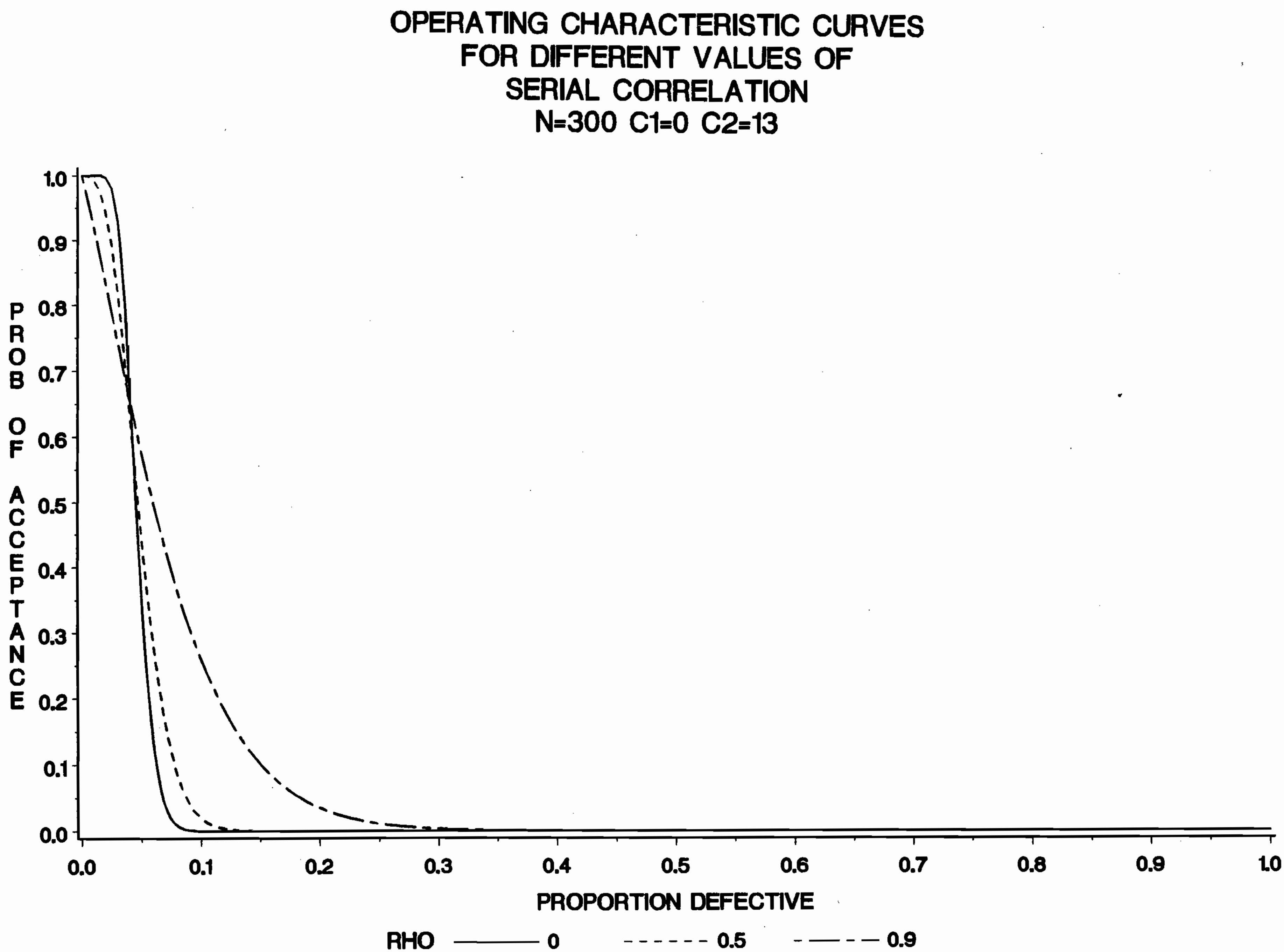
OPERATING CHARACTERISTIC CURVES
FOR DIFFERENT VALUES OF
SERIAL CORRELATION
N=300 C1=0 C2=13
PROB OF ACCEPTANCE
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
PROPORTION DEFECTIVE
RHO
0
0.5
0.9

Figure 2