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ABSTRACT

Single sampling plans used for lot acceptance are modified for use in monitoring production processes which exhibit Markov dependence in quality attributes. An augmented Markov chain model is employed for the determination of plan characteristics. Plans for dependent processes are compared with those for independent processes through numerical computations.

Key words: *Quality control of production processes, sampling plans Markov dependent processes, Markov chains.*

1. Introduction

The heightened awareness of the need for quality control of industrial output has increasingly drawn the attention of quality control scientists to efficient ways of monitoring production processes. Continuous sampling plans in their various incarnations (see Dodge (1943), Derman et al. (1957), Derman et al. (1969) and Sampathkumar (1984)) and quality control charts have been the major statistical tools employed for this purpose. Often in a production process, the quality of items is serially dependent (see for instance, Broadbent (1958)). With this in mind we propose here a simple sampling inspection procedure, which has the simplicity of acceptance sampling plans with single samples, with the additional time saving features of sequential sampling plans. The inspection procedure can also be used for acceptance sampling after production, if items are serially identifiable.

We assume that the quality of an item is an attribute which can be classified as good or bad with a Markovian dependence structure. The parameters used in specifying it are the long term fraction defective and the serial correlation coefficient. The proposed plan is as follows. After allowing for some time for the process to attain stability inspect a maximum of n ($\geq$ 2) items sequentially from a production run. If at any time the total number of defective items exceeds c ($\geq$ 0), the production is stopped for re-adjustment. Otherwise the process is continued until the run is complete. In the acceptance sampling vocabulary, a single production run is a lot, stopping the process is the rejection of the lot and continuing the process is acceptance. We shall use this terminology for convenience

in the rest of the paper when properties of this procedure are investigated.

In Section 2 we define an augmented Markov chain for the underlying process. It is used in section 3 to determine plan characteristics such as the probability of acceptance (i.e. continuing production), average outgoing quality and average sample size. In section 4, plan characteristics are illustrated using graphs based on numerical calculations. Finally, we discuss some issues pertaining to the design of sampling plans for Markov dependent production processes in the last section.

2. An Augmented Markov Chain

Let $\{X_n, n=0,1,2,\ldots\}$ be a two state Markov chain with states 0 and 1. We assume that the quality attributes of the nth inspected item can be represented by X_n, with states 0 and 1 representing good (acceptable) and bad (unacceptable) respectively. Let the transition probability p_{ij} be defined as

$$p_{ij} = P[X_{n+1}=j \mid X_n = i] \qquad n = 0,1,2,\ldots. \quad i,j = 0,1.$$

It is well known that the process is completely specified by the transition probability matrix

$$P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \qquad (1 \geqq a,\ b \geqq 0) \tag{1}$$

along with the initial state X_0.

Let

$$p_{ij}^{(n)} = P(X_n = j \mid X_0 = i). \qquad n = 1,2,\ldots \tag{2}$$

In a two state Markov chain defined as in (1), when $|1 - a - b| < 1$ the limiting probabilities

$$\pi_j = \lim_{n\to\infty} p_{ij}^{(n)} \quad , \qquad (j = 0,1)$$

exist independent of the initial state and are given by

$$(\pi_0,\pi_1) = \left(\frac{b}{a+b} , \frac{a}{a+b}\right) . \tag{3}$$

Also if we denote by ρ the serial correlation of the process for which the initial state X_0 has the distribution (3), we have

$$\rho = 1 - (a+b) . \tag{4}$$

Corresponding to the fraction defective in a production process with sequential items independent of each other (which we shall call henceforth "an independent production process"), in a Markov dependent process we may identify π_1, the probability that an item is defective in the long run. Thus a Markov dependent production process can be identified by two parameters, the fraction defective p $(= \pi_1)$ and the serial correlation ρ, with the following admissible ranges:

$$1 - \min\left\{\frac{1}{p}, \frac{1}{1-p}\right\} < \rho < 1$$

which can be stated also as

$$\max\left\{0, \frac{-\rho}{1-\rho}\right\} < p < \min\left\{1, \frac{1}{1-\rho}\right\} \quad . \tag{5}$$

Using p and ρ the transition probability matrix P can be represented

as

$$P = \begin{bmatrix} 1 - p(1-\rho) & p(1-\rho) \\ (1-p)(1-\rho) & p+\rho(1-p) \end{bmatrix} \quad . \tag{6}$$

Since sampling is to begin after the process has attained stability we may assume the initial state distribution to be $[(1-p),p]$. The sampling plan calls for inspecting a maximum of n ($\geq$ 2) items sequentially and stopping the production when the accumulated number of defectives exceeds c ($\geq$ 0). For the probability of acceptance (of the operating characteristic curve) therefore, we need the probability of finding c or less defectives in a sample of n Markov dependent observations. The distribution of the number of successes in a sequence of dependent trials given by Gabriel (1949) is not very convenient to determine this probability (also see, Nain and Sen (1980)). Instead, we proceed as follows.

Define an augmented bivariate Markov chain with the state vector (of two elements): cumulative number of defectives, present state. Thus when c defectives are allowed, the states are $\{00, 11, 10, 21, 20, \ldots, c1, c0, (c+1,1)\}$. The state (c+1,1) is absorbing and the production will be stopped (rejected) if the process enters this state. Note that the total number of states is 2c+2. The transition probability matrix for this process is given by

$$
R = \begin{array}{c}
\\ 00 \\ 11 \\ 10 \\ 21 \\ 20 \\ \cdot \\ \cdot \\ \cdot \\ c1 \\ c0 \\ c+1,1
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} 00 & 11 & 10 & 21 & 20 & 31 & \cdots & c1 & c0 & c+1,1 \end{array} \\
\begin{bmatrix}
1-a & a & & & & & & & & \\
 & & b & 1-b & & & & & & \\
 & & 1-a & a & & & & & & \\
 & & & & b & 1-b & & & & \\
 & & & & 1-a & a & & & & \\
 & & & & & \cdot & & & & \\
 & & & & & & \cdot & & & \\
 & & & & & & & \cdot & & \\
 & & & & & & & & b & 1-b \\
 & & & & & & & & 1-a & a \\
 & & & & & & & & & 1
\end{bmatrix}
\end{array}
\tag{7}
$$

This transition probability matrix plays a key role in the determination of the operating characteristics of the sampling plan. Probabilities of acceptance and rejection are obtained as elements of the powers of R which can be either determined by simple matrix multiplication or explicitly on account of the triangular structure of the matrix.

3. Plan Characteristics

In this section we give procedures for the determination of probabilities of acceptance and rejection, average outgoing quality and the average sample size.

Let $R^{(k)}{}_{ij}$ be the (i,j) element of the kth power of R given by (7). Using the properties of Markov chains, the probability of rejection of the process with a sample size $\leq$ k+1 is obtained by the last elements of the first two rows of R^k. Note that for acceptance (continuing production) all n items will have to be inspected, whereas rejection can come any time earlier when the number of defectives exceeds c.

Using the initial distribution $[(1-p),p]$, the probability of

acceptance P_a is obtained as

$$P_a = (1-p) \sum_{\ell,m \neq c+1,1} R^{(n-1)}_{00,\ell m} + p \sum_{\ell,m \neq c+1,1} R^{(n-1)}_{11,\ell m} \tag{8}$$

and the probability of rejection with a sample size k+1 or less is given by

$$P_r(k) = (1-p) R^{(k)}_{00,(c+1,1)} + p\, R^{(k)}_{11,(c+1,1)} \ . \tag{9}$$

Consequently the probability of rejection with a sample size k+1

$$P_r^{(k)} = P_r(k) - P_r(k-1) \ . \tag{10}$$

Since production is stopped (lot rejected) whenever the number of defectives reaches c+1, average sample size (ASN) is of interest. This can be obtained as

$$ASN = \sum_{k=c}^{n-2} (k+1) P_r^{(k)} + n\left(P_a + P_r^{(n-1)}\right) \ . \tag{11}$$

We assume that defective items found during inspection are discarded. Since production stops when the number of defectives exceeds c the average outgoing quality (AOQ) depends on the number of items accepted and the number of defectives in the accepted lots. We define

$$AOQ = \frac{E(\text{Number of defectives in accepted lots})}{E(\text{Number of items accepted})} \ . \tag{12}$$

Note that the denominator includes only good items in the stopped production runs. Using simple probability arguments we have

$$E[\text{Number items accepted}]$$

$$= [N - E(D)]P_a + \sum_{k=c}^{n-1} (k-c)\, P_r^{(k)} \tag{13}$$

where E(D) is the expected number of defectives found out of n in accepted lots. We have

$$E(D) = np \ . \tag{14}$$

$$E[\text{Number of defectives in accepted lots}]$$

$$= (N-n)pP_a \ . \tag{15}$$

We get

$$AOQ = \frac{(N-n)pP_a}{(N-np)P_a + \sum_{k=c}^{n-1} (k-c)P_r^{(k)}} \tag{16}$$

Ignoring the items produced in the stopped runs, we get an approximate result

$$AOQ \cong \frac{(N-n)p}{N-np} \ . \tag{17}$$

4. Numerical Results

In order to illustrate the properties of these sampling plans, we present several graphs based on numerical calculations. These graphs are grouped as follows. Note that when $\rho = 0$ we get the independent production process.

A. Probability of acceptance vs. Serial correlation

n=80, c=3 for p = .01, .03, .05, .2.

(Figs. A1-A4)

B. Operating characteristic curves

$\underline{\rho = -0.05,\ 0,\ 0.5}$

n=10, c=1
n=20, c=2
n=40, c=4
} (Figs. B1-B3)

$\underline{\rho = -0.5,\ 0,\ 0.5}$

n=20, c=3
n=40, c=3
n=80, c=3
} (Figs. B4-B6)

$\underline{\rho = -0.05,\ -0.02}$

n=80, c=3
} (Fig. B7)

$\underline{c = 2,3,4}$

n=40, ρ=0
n=40, ρ=0.5
n=80, ρ=0
n=80, ρ=0.5
} (Figs. B8-B11)

C. ASN curves

$\underline{\rho = -0.5,\ 0,\ 0.5}$

n=40, c=3
n=80, c=3
} (Figs. B12-B13)

Based on these graphs the following observations are in order.

A. For a given plan and a fraction defective p, the acceptance probability is convex in ρ.

B. (1) Larger the sample size, better the discriminating power of the plan. (Figs. B1-B3)

(2) Smaller the value of ρ, better is the discriminating power of the plan (Figs. B1-B7). Note that when ρ is

negative the range of admissible value of p is restricted (see (5)).

(3) Conclusion (2) is preserved for different acceptance numbers as well. (Figs. B8-B11)

(4) For some ranges of values of p, negative ρ values result in smaller average sample sizes. When ρ is positive, whether the average sample size is smaller or larger than the independent process case depends on the range of p values. For smaller p values independent process gives larger sample sizes whereas for larger p values, it gives smaller sample sizes. (Fig. B13, B14)

5. Design Issues

Designing a sampling plan (i.e. to determine sample size n and the acceptance number c) for given values of (AQL, producer's risk α) and (LTPD, consumer's risk β) becomes difficult because of the complicated form of the probability of acceptance. Even though we have used sample matrix multiplication techniques in obtaining numerical values, as shown in Bhat and Lal (1987) explicit expressions can also be obtained for the elements of the powers of the augmented matrix R, leading to explicit expressions for the probability of acceptance. Nevertheless, the expressions are non-linear in n and c and therefore a search technique seems more appropriate.

The search for the best (n,c) pair can be carried out by first fixing n and then choosing c which gives the best protection for the producer and the consumer. The following tables provide indications of issues that need to be taken into consideration in this decision.

The following notations are used: ρ = serial correlation, n = sample size; c = acceptance number; p_0 = AQL; α = producer's risk; p_1 = LTPD; β = consumer's risk. For ease of understanding the probabilities have been rounded off to two decimals in most of the cases. When ρ is negative the fraction defective p can assume only a restricted range of values.

$\rho = -.03$

	n = 40				n = 80			
c	p_0	α	p_1	β	p_0	α	p_1	β
2			.10	.22			.10	.01
	.03	.10	.15	.04	.03	.42	.15	.0002
3			.10	.42			.10	.03
	.03	.02	.15	.13	.03	.20	.15	.001
4			.10	.64			.10	.09
	.03	.004	.15	.26	.03	.08	.15	.004

$\rho = -.02$

	n = 40				n = 80			
c	p_0	α	p_1	β	p_0	α	p_1	β
2	.02	.04	.10	.22	.02	.21	.10	.01
	.03	.11	.15	.05	.03	.43	.15	.0002
3	.02	.01	.10	.42	.02	.07	.10	.03
	.03	.03	.15	.13	.03	.21	.15	.001
4	.02	.001	.10	.63	.02	.02	.10	.09
	.03	.01	.15	.20	.03	.08	.15	.004

$\rho = 0$

	n = 40				n = 80			
c	p_0	α	p_1	β	p_0	α	p_1	β
2	.02	.05	.10	.22	.02	.22	.10	.01
	.03	.12	.15	.05	.03	.43	.15	.0003
3	.02	.01	.10	.42	.02	.08	.10	.04
	.03	.03	.15	.13	.03	.22	.15	.001
4	.02	.001	.10	.63	.02	.02	.10	.09
	.03	.01	.15	.26	.03	.09	.15	.005

$\rho = .5$

c	n = 40				n = 80			
	p_0	α	p_1	β	p_0	α	p_1	β
2	.02	.12	.10	.38	.02	.25	.10	.10
	.03	.18	.15	.19	.03	.38	.15	.02
3	.02	.07	.10	.51	.02	.16	.10	.16
	.03	.11	.15	.29	.03	.27	.15	.04
4	.02	.04	.10	.63	.02	.10	.10	.24
	.03	.07	.15	.40	.03	.18	.15	.07

Suppose we are looking for a plan with the protection

$p_0 = .02$ $\qquad$ $p_1 = .15$

$\alpha = .05$ $\qquad$ $\beta = .05$

For n = 40, c = 2 is appropriate when $\rho = 0$. This plan is adequate for negative ρ values closer to 0. But when $\rho = .5$, c=2 gives $p_0 = .02$, $\alpha = .12$ and $p_1 = .15$, $\beta = .19$. Nevertheless, of the options presented (n=40, c=2) still seems to be the best option.

A significant conclusion seems to be that when the serial correlation of a Markov dependent production process is larger, single sampling plans designed for an independent process do not provide the same kind of protection for the producer or the consumer.

REFERENCES

U. Narayan Bhat and Ram Lal (1987). "Some Results for Correlated Random Walks" forthcoming.

Broadbent, S. R. (1958). "The Inspection of a Markov Process", J. Roy. Stat. Soc. B20, 111-119.

Derman, C., Johns, M.V. and Lieberman, G. J. (1969). "Continuous Sampling Procedures Without Control", Ann. Math. Statist. 30, 1175-1191.

Derman, C., Littauer, S., and Solomon, H. (1957), "Tightened Multilevel Continuous Sampling Plans", Ann. Math. Statist. 28, 395-404.

Dodge, H. F. (1943), "A Sampling Inspection Plan for Continuous Production", Ann. Math. Statist. 14, 264-279.

Gabriel, K. R. (1959), "The Distribution of the Number of Successes in a Sequence of Dependent Trials", Biometrika 46, 454-460.

Nain, R. B. and Sen, Kanwar (1980), "Transition Probability Matrices for Correlated Random Walks", J. Appl. Prob. 17, 253-258.

Sampathkumar, V.S. (1984). "A Tightened m-Level Continuous Sampling Plan for Markov Dependent Production Processes," IEEE Trans. 16, 257-261.

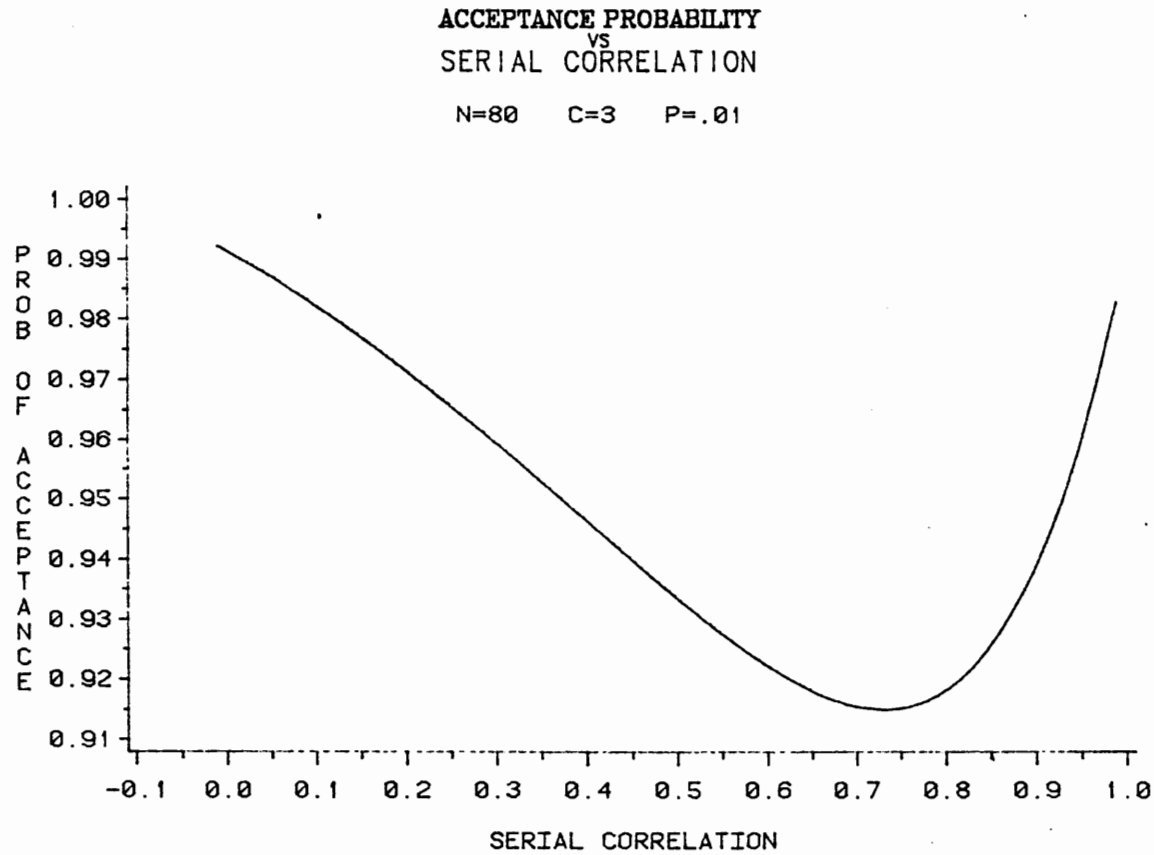


FIGURE A1

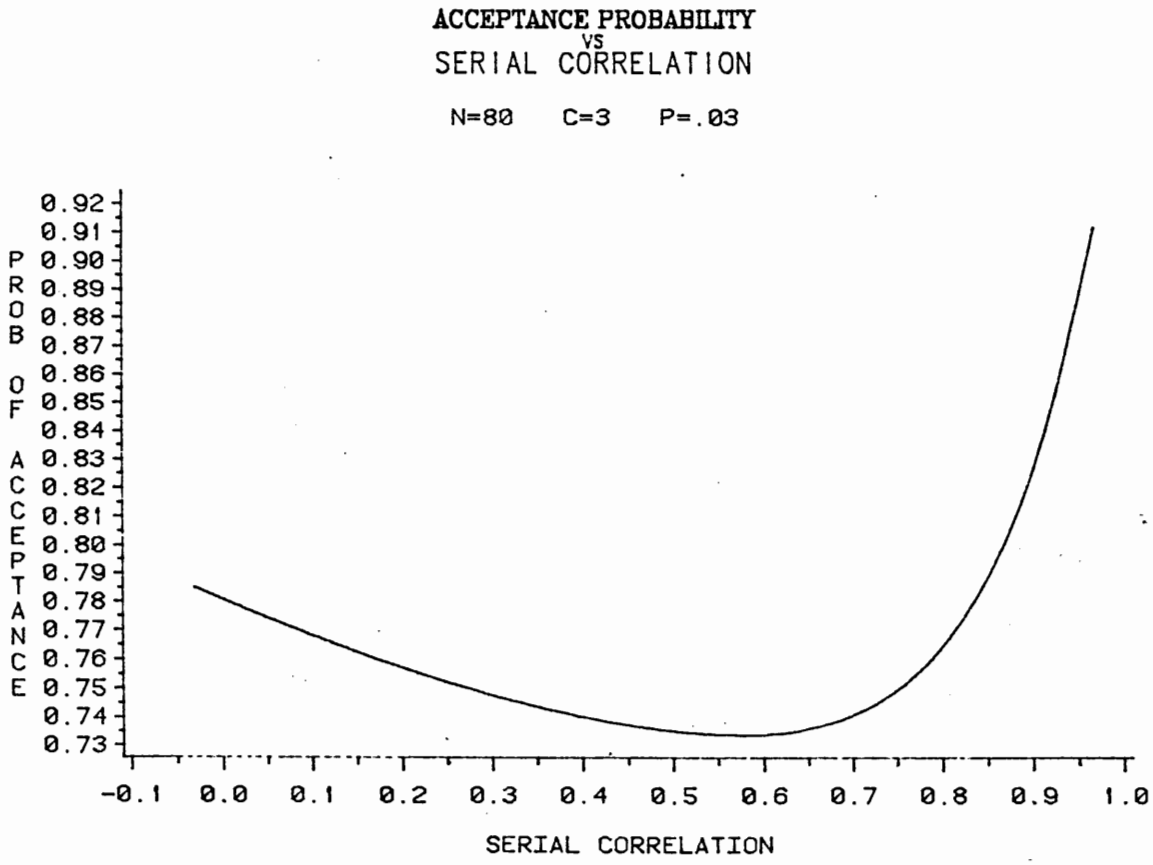


FIGURE A2

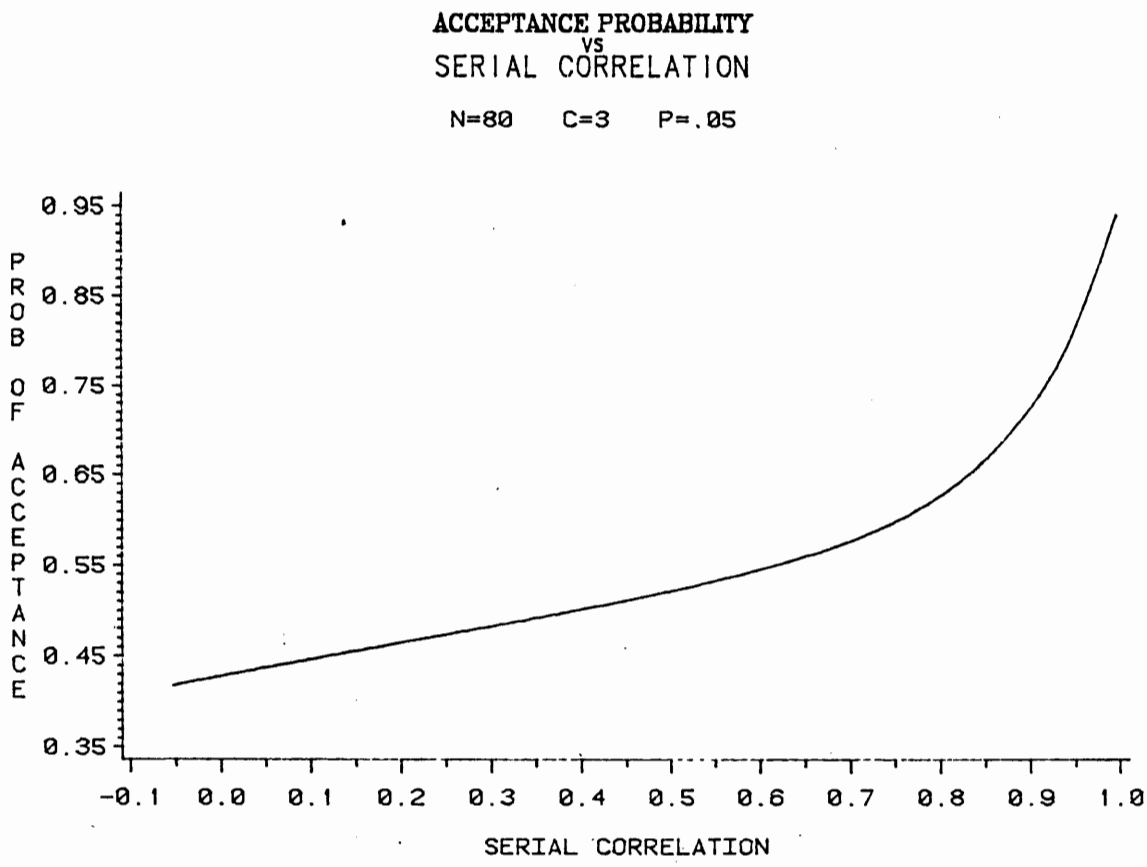


FIGURE A3

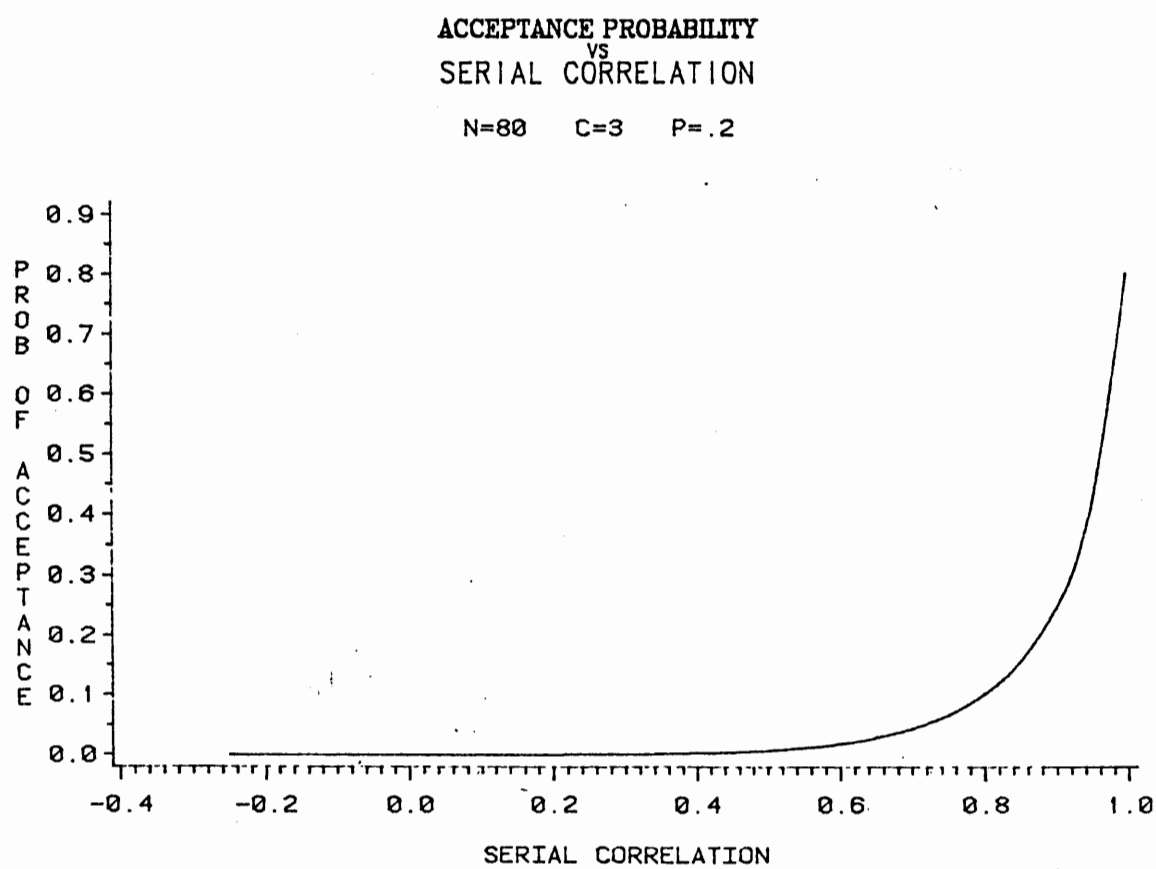


FIGURE A4

OPERATING CHARACTERISTIC CURVES FOR THE DIFFERENT VALUES OF SERIAL CORRELATION

N=10 C=1

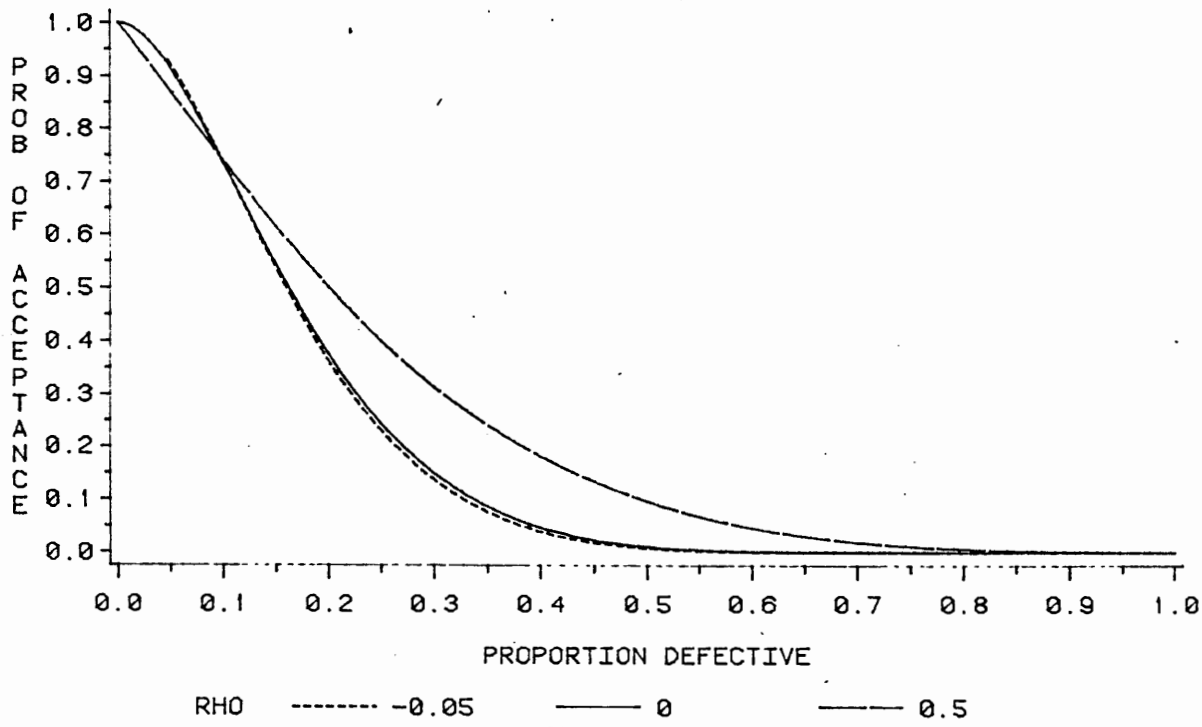


FIGURE B1

OPERATING CHARACTERISTIC CURVES FOR THE DIFFERENT VALUES OF SERIAL CORRELATION

N=20 C=2

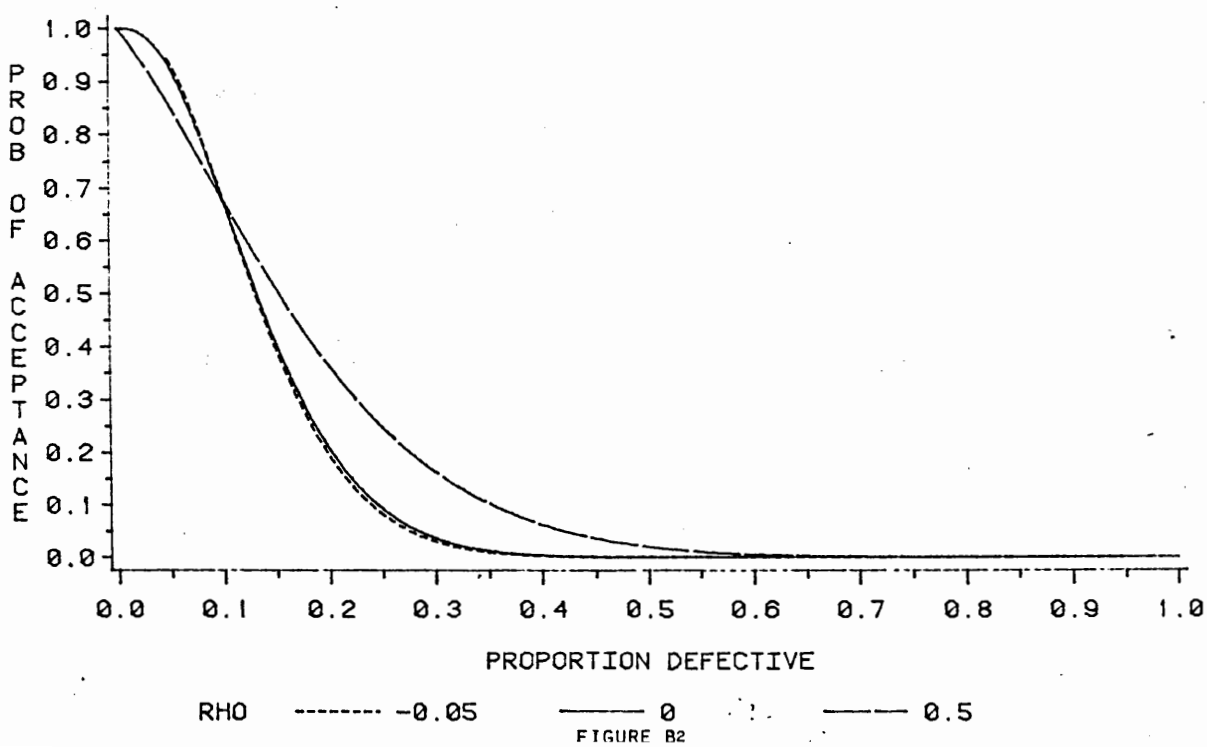


FIGURE B2

OPERATING CHARACTERISTIC CURVES FOR THE DIFFERENT VALUES OF SERIAL CORRELATION

N=40 C=4

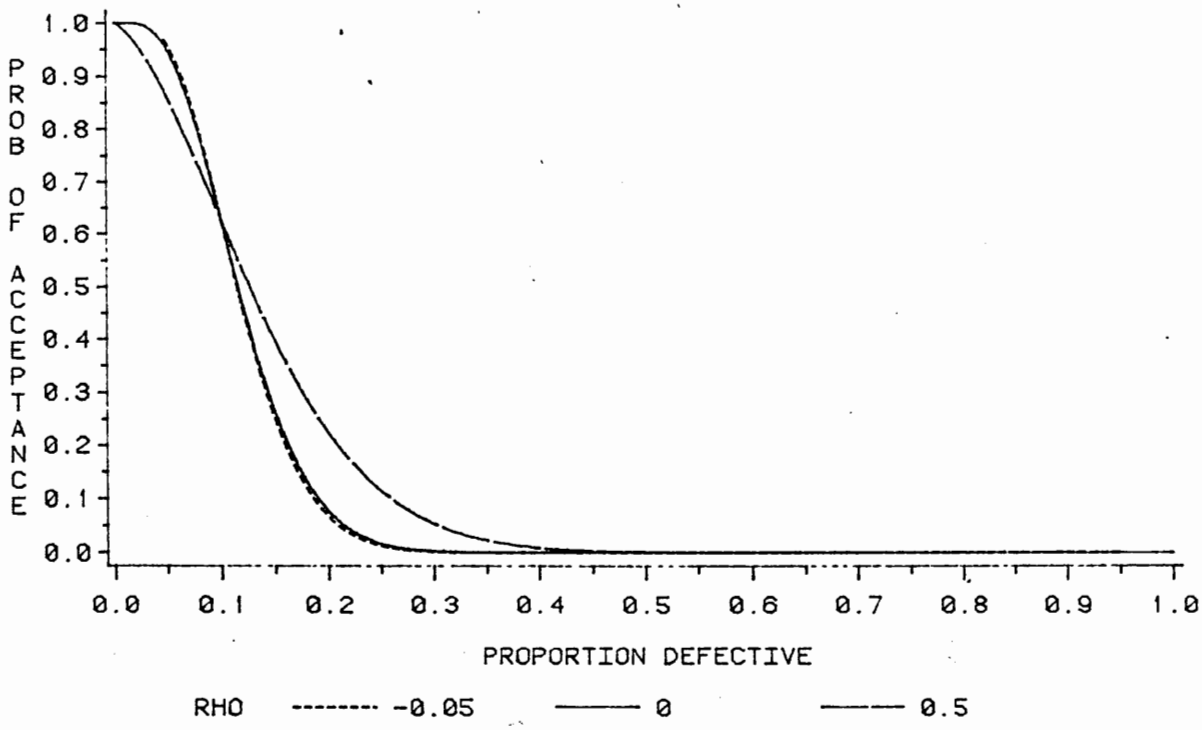
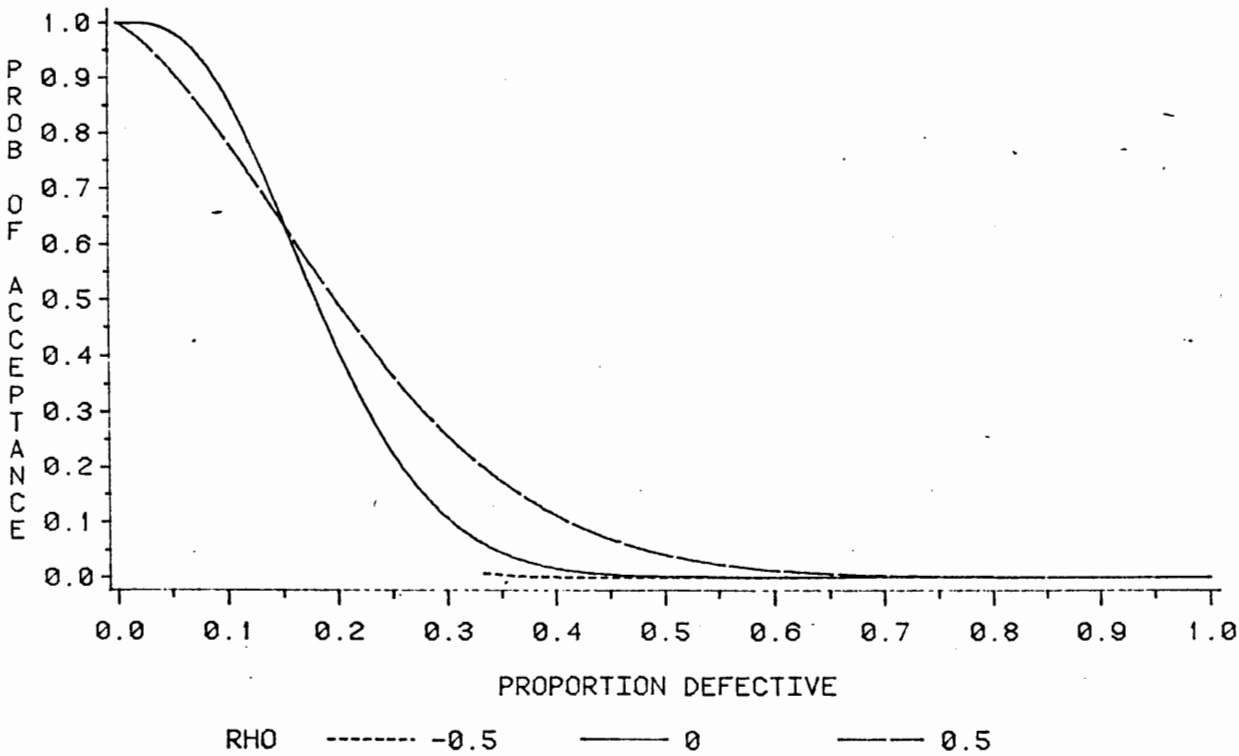


FIGURE B3

OPERATING CHARACTERISTIC CURVES FOR THE DIFFERENT VALUES OF SERIAL CORRELATION

N=20 C=3



OPERATING CHARACTERISTIC CURVES FOR THE DIFFERENT VALUES OF SERIAL CORRELATION

N=40 C=3

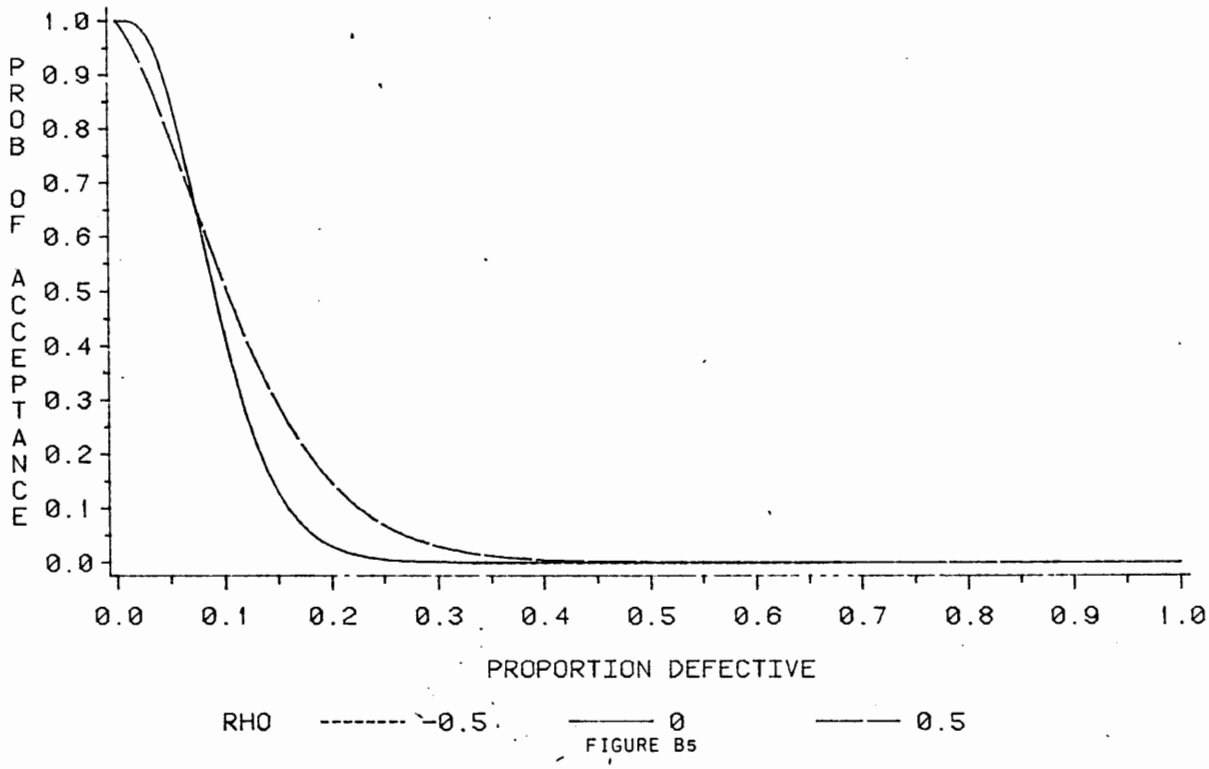
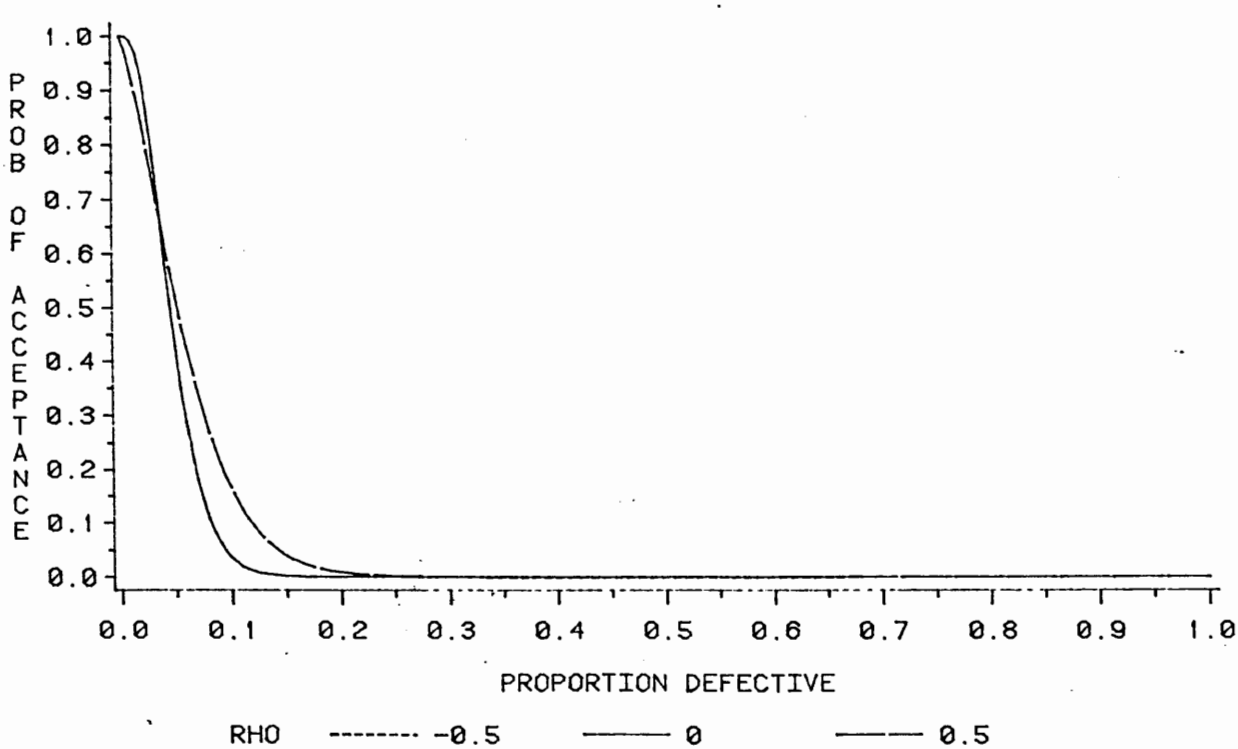


FIGURE B5

OPERATING CHARACTERISTIC CURVES FOR THE DIFFERENT VALUES OF SERIAL CORRELATION

N=80 C=3



OPERATING CHARACTERISTIC CURVES FOR THE DIFFERENT VALUES OF SERIAL CORRELATION

N=80 C=3

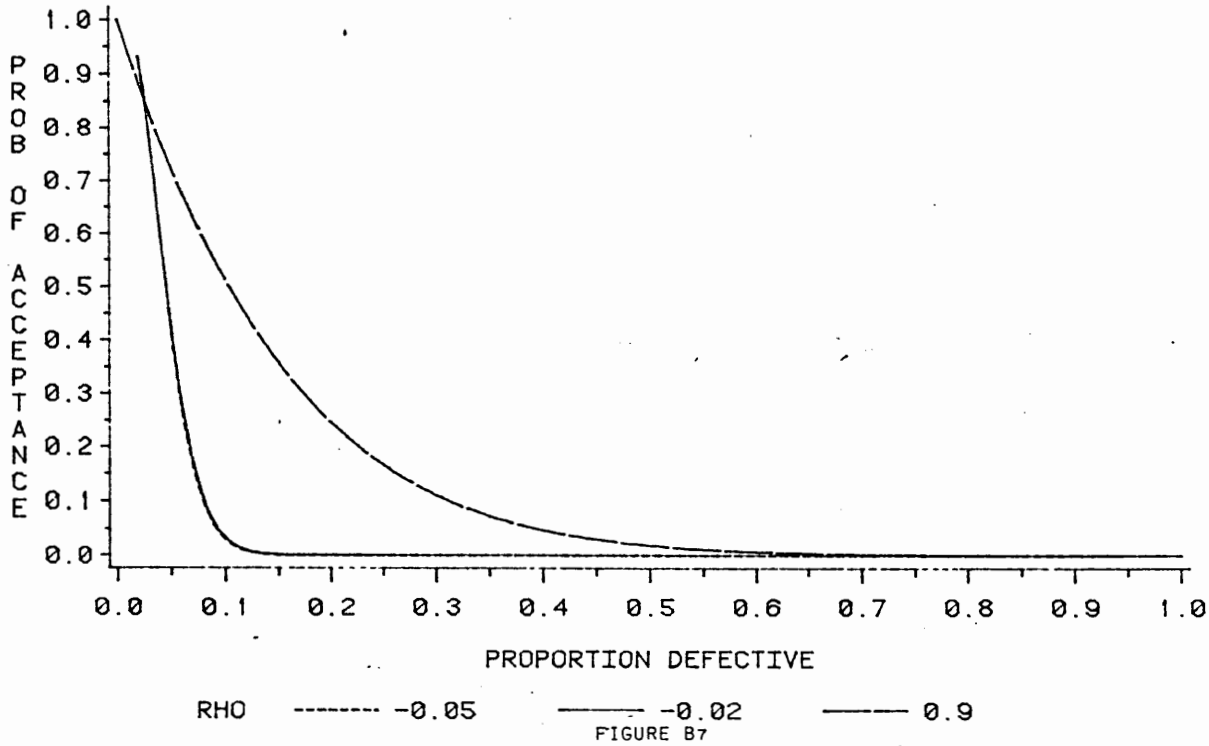


FIGURE B7

OPERATING CHARACTERISTIC CURVES FOR DIFFERENT C VALUES

N=40 RHO=0

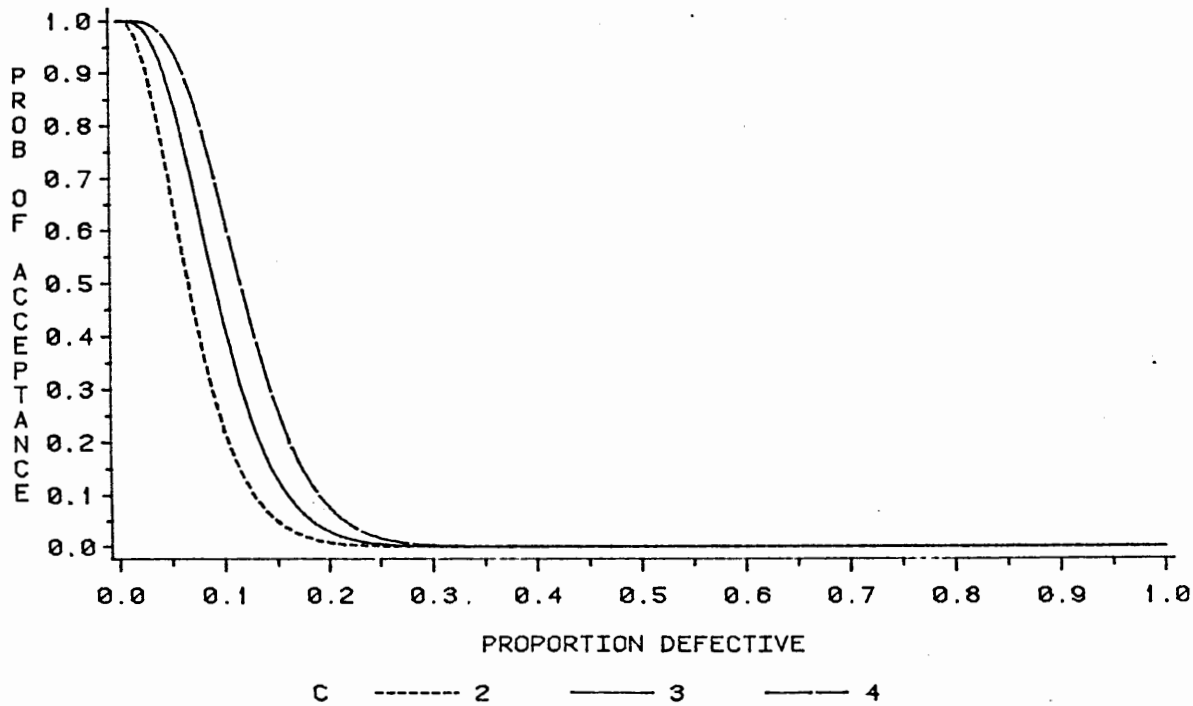


FIGURE B8

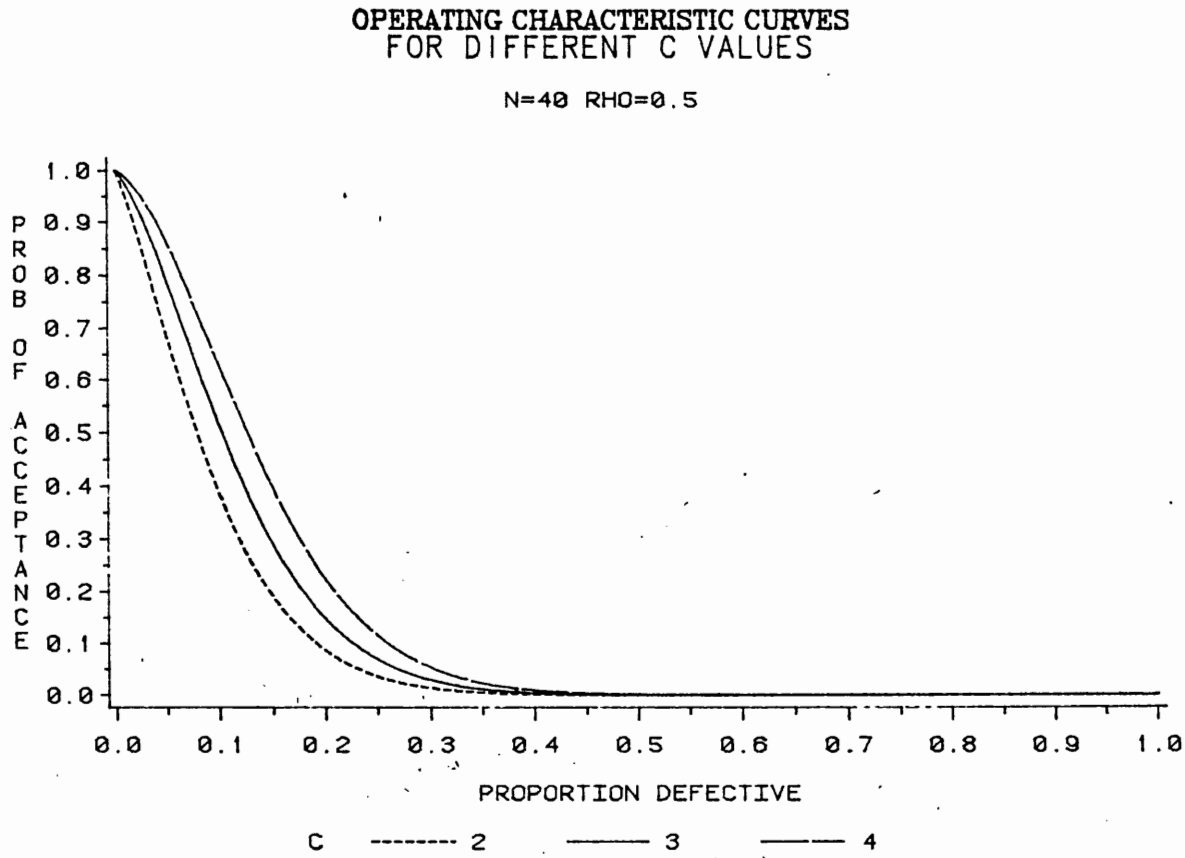


FIGURE B9

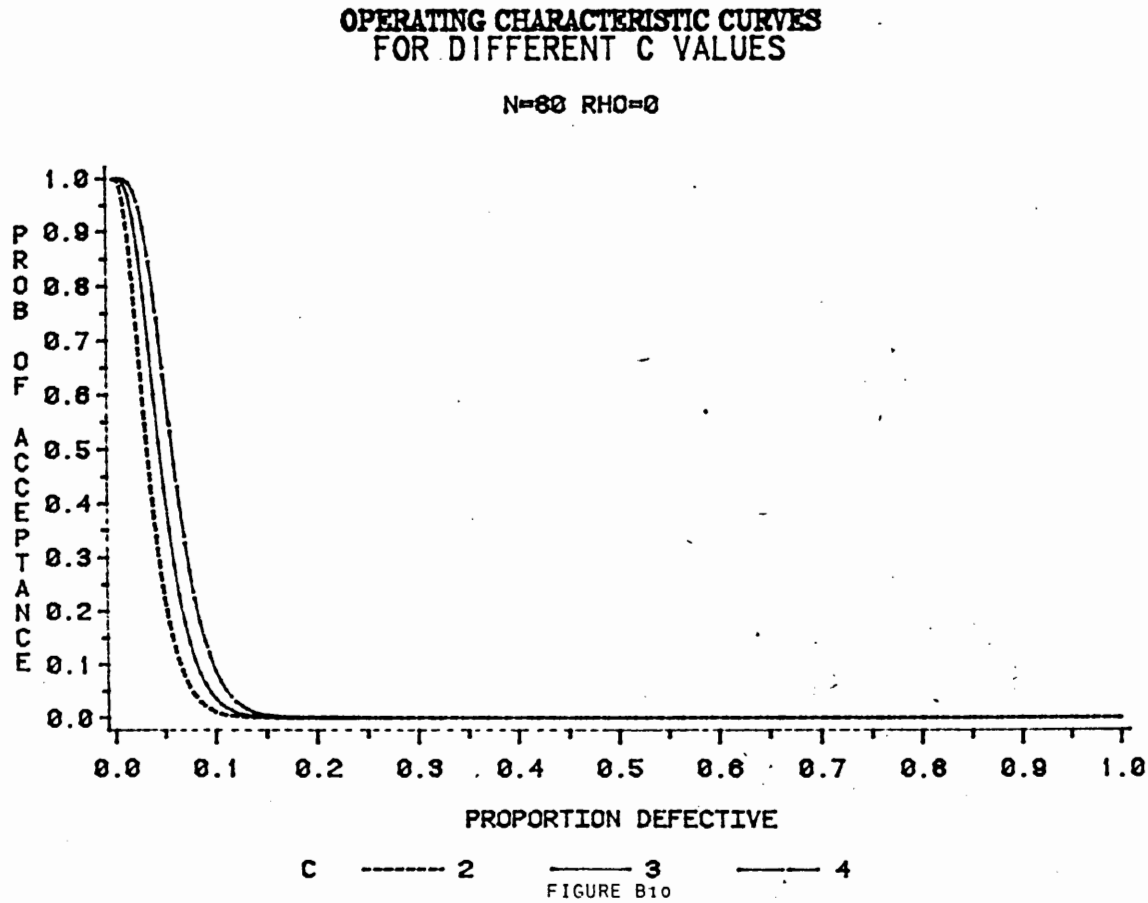


FIGURE B10

OPERATING CHARACTERISTIC CURVES
FOR DIFFERENT C VALUES

N=80 RHO=0.5

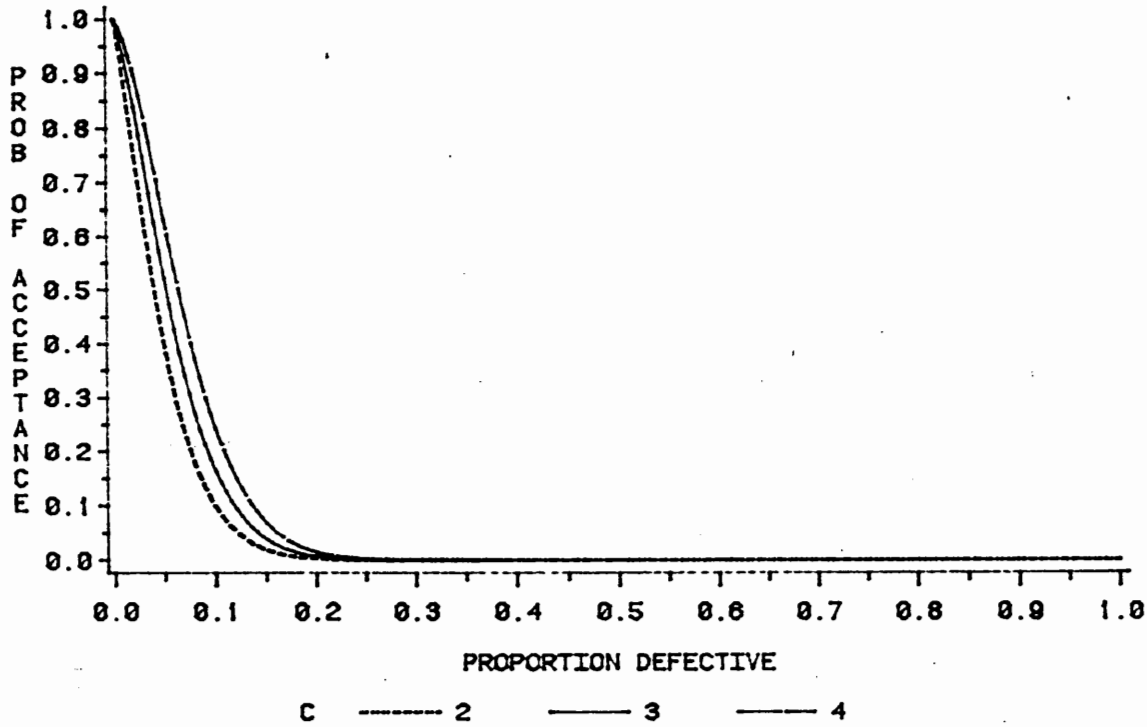


FIGURE B11

ASN CURVES
FOR THE DIFFERENT VALUES OF RHO

N=40 C=3

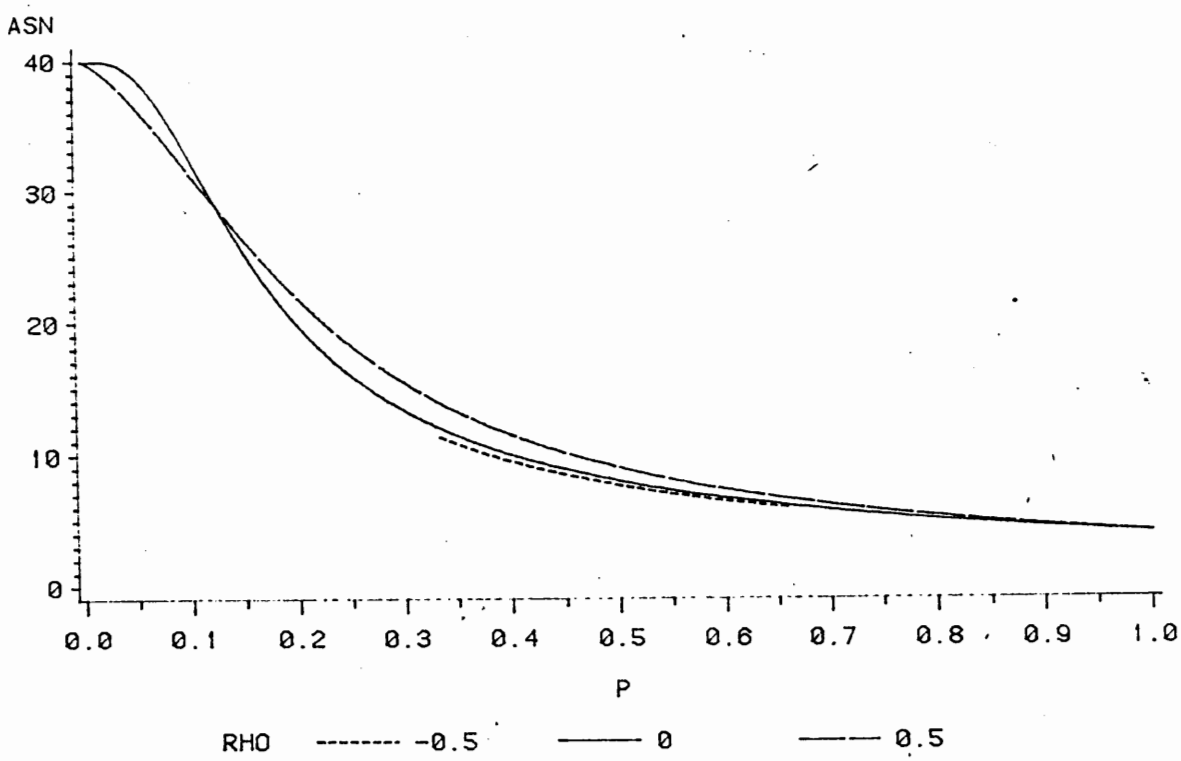


FIGURE B12

ASN CURVES
FOR THE DIFFERENT VALUES OF RHO

N=80 C=3

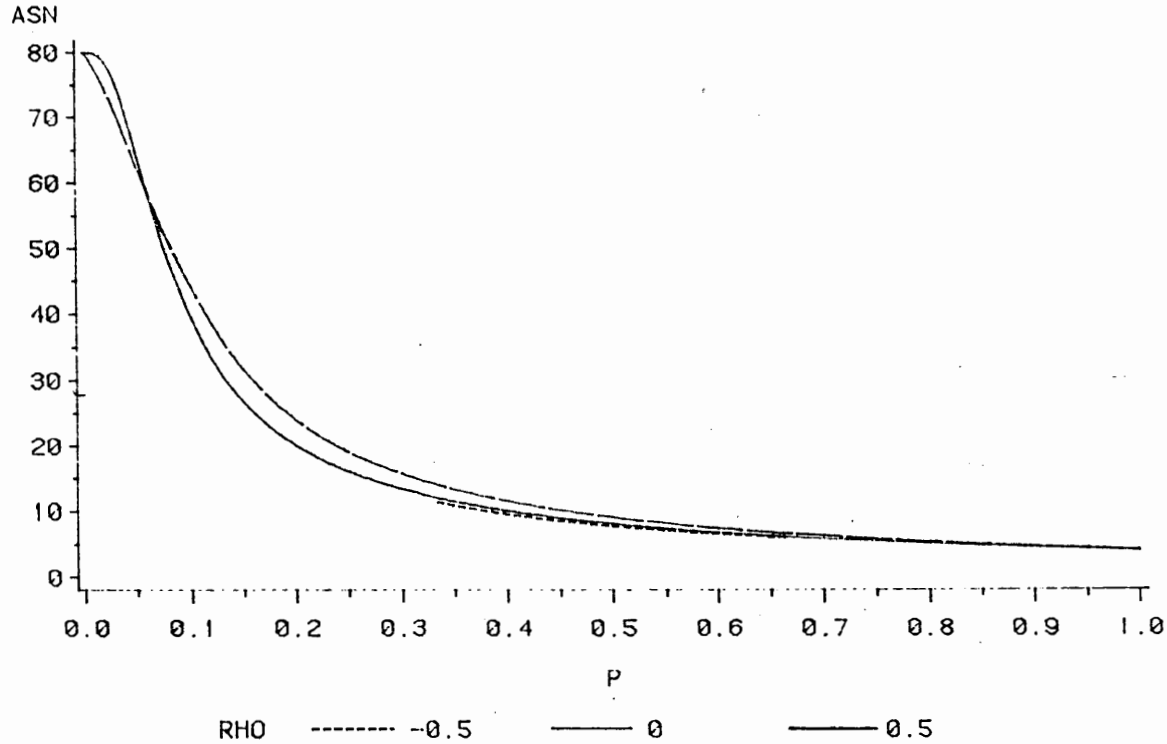


FIGURE B13