THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

PROBABILITY OF LOCATING A SUBMARINE WITHIN A STATED DISTANCE BY USE OF A RANGE-BEARING SENSOR

by

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Technical Report No. 2
Department of Statistics THEMIS Contract

September 3, 1968

Research sponsored by the Office of Naval Research
Contract N00014-68-A-0515
Project NR 042-260

DEPARTMENT OF STATISTICS
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ABSTRACT

The ocean surface is considered to be a geometrical plane, and locations under the surface are points of a three-dimensional rectangular coordinate system. The general problem is to estimate the surface position beneath which a submarine is located. Estimation is based on the observed submarine range, and its observed horizontal bearing, as furnished by a range-bearing sensor that is at a determined location in the ocean. This information, combined with an often reasonable assumption about the true vertical bearing, yields an approximate estimate. The specific problem is to evaluate the probability that the true (surface) location of the submarine is within a stated distance of the estimated location. An approximate expression is developed for the value of this probability, subject to some assumptions and simplifications.

Introduction

Geometrically, the ocean surface is considered to be a plane and locations under the surface are represented by points of a three-dimensional rectangular coordinate system (in which the third coordinate provides depth). The general problem is to estimate the position on the ocean surface below which the center of the submarine (referred to as the submarine position) is located. Information for this estimation is furnished by a range-bearing sensor which is located at a determined position in the ocean. That is, with respect to its own location, the sensor provides an observed distance and an observed horizontal direction to the submarine position at a single specified time. Also, as a parameter whose value is ordinarily unknown, the vertical angle (with respect to the horizontal) of the

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*Based on work done while the author was with Lockheed Aircraft Corporation and System Development Corporation. Written in association with Office of Naval Research Contract No. N00014-68-A-0515.

submarine from the sensor position is included in the information for the analysis. This totality of information, combined with an often reasonable assumption about the true vertical angle, yields an estimated horizontal location for the submarine.

Consider the ocean surface and let a circle with given radius be centered at the estimated position of the submarine (position under which the submarine is located). The specific problem is to approximately determine the probability that the true position of the submarine is contained in this circle. This probability depends on the vertical angle of the submarine from the sensor, the radius of the circle, the probability distribution of the errors for estimation of horizontal bearing, and the probability distribution of the errors for estimation of range. The purpose of this analysis is to identify the parameters that arise and, in terms of these parameters, develop an approximate expression for the probability that the true submarine position is contained in the circle.

Assumptions and Simplifications

1. The ocean surface is considered to be flat, and depth represents a direction that is perpendicular to the ocean surface.

2. The sensor can be represented as a point, and the (surface) location of this point is exactly known at the time considered.

3. Use of the submarine center to represent the submarine position does not result in any important inaccuracies.

4. Time coordination is such that the readings of the submarine range and bearing, with respect to the sensor, correspond to the submarine location at the time considered.

5. The observed bearing and the observed range provided by the sensor are statistically independent.

6. The observed bearing has a normal (Gaussian) probability distribution with mean equal to the true horizontal direction of the submarine position.

7. The observed range has a normal (Gaussian) probability distribution with mean equal to the true distance of the submarine from the sensor.

8. The standard deviation of the observed bearing is small (say, at most 0.03 radians).

9. The standard deviation of the observed range is small compared to the true value of the range (say, at most 3 percent of the true range).

10. Assumptions 8 and 9 are considered to imply that second and higher order terms in angular errors and/or range errors can be neglected in derivations.

Notation and Relationships

Strictly speaking, the situation is of a three-dimensional nature. However, the pertinent analysis can be performed in a two-dimensional framework for many of the situations of importance.

Specifically, consider the ocean surface and suppose that the sensor is located under (or at) the origin of the x,y-rectangular coordinate system that is used for the surface. As additional standardization, suppose that the submarine position is under the point $(D\cos\Theta, 0)$, where D is the distance from the (three-dimensional) sensor location to the (three-dimensional) submarine location, and Θ is the vertical angle, with respect to the horizontal, of the submarine position from the sensor position. Here, the horizontal angle of the submarine position, with respect to the positive x-axis, is taken to be zero radians (with no loss of generality).

The observed distance of the submarine from the sensor is D + d, and the observed horizontal angle of the submarine from the sensor is $\underline{a}$ radians. Here, the random errors d and $\underline{a}$ are statistically independent, $Ed = Ea = 0$, and σ_d is the (known) standard deviation of d while σ_a is the (known) standard deviation of $\underline{a}$.

Evaluation of Probability

Let

$$v = \min(\sigma_d^2, D^2\sigma_a^2), \qquad V = \max(\sigma_d^2, D^2\sigma_a^2).$$

Then, the probability to be evaluated can be expressed as

$$P[t_1^2 + Vt_2^2 \leq R^2/v(\cos\Theta)^2],$$

where t_1 and t_2 are independent random variables that have standardized normal probability distributions. Probabilities of this form can be approximately evaluated by the method given in ref. 1.

The cumulative distribution function of $t_1^2 + Vt_2^2$ can be expressed as

$$\sum_{w=0}^{\infty} H_w C_{2w+2}[x],$$

where $C_{2w+2}[x]$ is the χ^2 cumulative distribution function with 2w + 2 degrees of freedom. The H_w are nonnegative and add to unity. The values of the H_w are determined by

$$V^{-1/2}[1 - (1 - 1/V)z]^{-1/2} \equiv \sum_{w=0}^{\infty} H_w z^w,$$

for the expansion that occurs when $|z|$ is sufficiently small. For example,

$$H_0 = V^{-1/2}, \qquad H_1 = \tfrac{1}{2}(1 - 1/V)V^{-1/2}, \qquad H_2 = (3/8)(1 - 1/V)^2V^{-1/2},$$

$$H_3 = (5/16)(1 - 1/V)^3V^{-1/2}, \qquad H_4 = (35/128)(1 - 1/V)^4V^{-1/2}, \text{ etc.}$$

To obtain workable approximations, the following inequalities are helpful:

$$\sum_{w=0}^{W} H_w C_{2w+2}[x] \leq \sum_{w=0}^{\infty} H_w C_{2w+2}[x]$$

$$\leq \sum_{w=0}^{W} H_w C_{2w+2}[x] + \left[1 - \sum_{w=0}^{W} H_w\right] C_{2W+4}[x],$$

$(W \geq 0)$. Use of these inequalities, with $x = R^2/v(\cos\Theta)^2$, provides upper and lower bounds for $P[t_1^2 + Vt_2^2 \leq R^2/v(\cos\Theta)^2]$. By appropriate choice of W, this probability can be evaluated to reasonable accuracy without the use of very many terms.

Practical Use of Results

The estimate (X,Y), and the value of the probability, depend on the true (but usually unknown) value of $\cos\Theta$. A compromise value can be assigned to $\cos\Theta$. This compromise value is at most one percent in error for a moderately wide range of values for Θ. Specifically, let $\cos\Theta$ equal 0.99. Then, there is an error of at most one percent (approximately) if the true value of Θ lies between - 0.2 and 0.2 radians (between - 11.5 and 11.5 degrees). If the depth of the sensor is suitably chosen, and the distance to the submarine is not too small, use of $\cos\theta = 0.99$ should provide reasonably accurate results.

REFERENCE

1. Herbert Robbins and E. J. G. Pitman, "Application of the method of mixtures to quadratic forms in normal variates," Annals of Mathematical Statistics, Vol. 20 (1949), pp. 552-560.