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DEPARTMENT OF STATISTICS
Southern Methodist University
Dallas, Texas 75725

On Wolfe's Test for Related Correlation Coefficients

John E. Boyer, Jr and William R. Schucany
Department of Statistics
Southern Methodist University
Dallas, Texas 75275

Abstract

The comparison of the strength of association between a variable, X_1, and each of two potential linear predictors, X_2 and X_3, is reexamined. The variances of X_2 and X_3 are nuisance parameters, which must be assumed to be equal in the procedure recently suggested by Wolfe [11]. In this note a simple modification of Wolfe's test is proposed. The use of ranks allows one to avoid the scale problem.

Key words: Rank correlation, Unequal variances, Association, Normal scores

1. Introduction

Let (X_{1i}, X_{2i}, X_{3i}) $i = 1, \ldots, n$ be a random sample of observations from a continuous trivariate distribution. In many situations, we are interested in determining which of X_2 and X_3 is more strongly correlated with X_1.

Wolfe [10] showed that if $\mathrm{Var}(X_2) = \mathrm{Var}(X_3)$ then the correlation between X_1 and X_2 is equal to that between X_1 and X_3 if and only if X_1 and $Z = X_3 - X_2$ are uncorrelated. Subsequently, the same author [11] proposed a distribution-free procedure for detecting a difference between the two correlations.

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The procedure was exemplified for a set of heart disease data with the significant indication that X_3 is more positively related to X_1 than is X_2.

Examination of the data suggests otherwise, since r_{12} is greater than r_{13}, where r_{ij} is the sample product moment correlation coefficient between X_i and X_j. The problem is that the equality of variance assumption evidently does not hold for these data and that this assumption is clearly crucial in the inferential process.

In Section 3 an alternative procedure is proposed which eliminates the scale problem. The method is applied to the same heart disease data from [11] and quite a different conclusion is reached.

2. Related Correlation Coefficients

Let X_1, X_2, X_3 have a continuous trivariate distribution with covariance matrix Σ. Let $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$ be the $(ij)^{th}$ element of Σ, with $\rho_{ii} = 1$ $(i,j = 1,2,3)$. For the trivariate normal the problem of testing H_o: $\rho_{12} = \rho_{13}$ has been discussed by Hotelling [7] and more recently by Dunn and Clark [3,4]. A distribution free approach relies upon observations by Wolfe [10] that the correlation between X_1 and Z is given by

$$\rho_{1z} = \frac{\rho_{13}\sigma_1\sigma_3 - \rho_{12}\sigma_2\sigma_1}{\sigma_1(\sigma_2^2 + \sigma_3^2 - 2\rho_{23}\sigma_2\sigma_3)^{1/2}} \tag{1}$$

and thus $\rho_{1z} = 0$ implies $\sigma_{12} = \sigma_{13}$, and in fact $\rho_{12} = \rho_{13}$ follows only if $\sigma_2 = \sigma_3$. This restriction on σ_2 and σ_3 is also necessary for Kendall's $\tau_{1z} > 0$ to reasonably imply that $\rho_{13} > \rho_{12}$. Consider the special case of joint normality for which $\tau = 2/\pi \arcsin(\rho)$.

Thus, from (1) and the fact that $\tau = 0$ if and only if $\rho = 0$, we see that $\tau_{1z} = 0$ does not imply either $\tau_{13} = \tau_{12}$ or $\rho_{13} = \rho_{12}$, but only that $\sigma_{12} = \sigma_{13}$. Similarly, $\tau_{1z} > 0$ does not preclude either $\tau_{13} < \tau_{12}$ or $\rho_{13} < \rho_{12}$.

The assumption of equal variances for X_2 and X_3 is suspect for the data analyzed by Wolfe [11]. A sample value T_{1z} of .35 is obtained, and viewed as a significant indication that X_3 is more positively related to X_1 than is X_2. The data do not support that conclusion, when the measure of "positively related" is any of the standard measures of association. For instance $r_{12} = .673$ but $r_{13} = .511$ and $T_{12} = .558$ but $T_{13} = .400$. The sample covariances, $s_{12} = 280.56$ and $s_{13} = 939.33$, are certainly consistent with the inference that $\sigma_{13} > \sigma_{12}$. However, the magnitudes of the sample variances, $s_2^2 = 4.25$ and $s_3^2 = 82.56$, suggest that it is not unreasonable for the sense of the inequality to be reversed for ρ_{13} and ρ_{12}.

3. Procedure for Unequal Variance

A number of modified procedures are available, all in the spirit of the method put forth by Wolfe. One approach involves scoring the X_{2i} and X_{3i} with an order preserving transformation that will circumvent the scale problem. Replacing each of the $\underline{X}_2$ and $\underline{X}_3$ vectors by their integer ranks, $R(X_{2i})$ and $R(X_{3i})$, is one possibility. The problem that arises here is that the $Z_i' = R(X_{3i}) - R(X_{2i})$ would necessarily involve a substantial number of ties. Replacing X_{2i} and X_{3i} by their expected normal scores will reduce the magnitude of this problem and still eliminate the scale problem.

Therefore, the suggested procedure is to replace the X_{2i} and X_{3i} by the corresponding expected normal scores, $a_i = E[Z_{(i,n)}]$, and use any of the usual nonparametric measures of correlation to detect association between the X_{1i} and the differences of the scores, $Z_i' = a(X_{3i}) - a(X_{2i})$. Because there is not a familiar population quantity (in terms of the original parameters) corresponding to the relationship between X_1 and Z', we shall not attempt to state formal hypotheses. Nevertheless, the technique allows one to make general inferences about the relationships of X_1 with X_2 and X_3. Note that this same difficulty accompanied the technique proposed by Wolfe [11]. Using Wolfe's test one is able to infer that X_1 and Z are positively (negatively) related. The difficulty arises in extending the knowledge of the relationship between X_1 and Z, to knowledge about the strength of the relationship between X_1 and X_2 relative to that between X_1 and X_3.

Since ties within the X_{2i} and X_{3i} are a problem that may be encountered, a consistent method for dealing with them is proposed. Averaging the scores for the tied values and then rescaling by a function of the total sum of squares is a scheme which both remains consistent with the midranking procedure, and maintains the equal variance property. If the normal scores are denoted by $\{a_i\}$ and the midranked set by $\{a_i^*\}$ then the proposed scores are given by $a_i' = a_i^*[\Sigma a_i^2 / \Sigma a_i^{*2}]^{1/2}$. Table 1 reflects the application of this rule due to ties among the X_{2i} and X_{3i}. If there are no ties then $a_i' \equiv a_i$ and the equal variance requirement is satisfied. The scores $\{a_i\}$ and their sum of squares are tabled in several places, e.g. Owen [9] p.151.

Table 1. Heart Disease Data

Country	X_{1i}	X_{2i}	X_{3i}	a'_{2i}	a'_{3i}	Z'_i
Australia	649.2	8	33	1.3781	.3169	-1.061
Canada	631.6	8	38	1.3781	.8822	-.496
Ceylon	173.7	2	17	-1.8068	-1.2928	.514
Chile	603.0	4	20	-.4092	-.9964	-.587
Denmark	330.6	6	39	.0797	1.5349	1.455
Finland	757.1	7	30	.5898	.0778	-.512
France	282.1	7	29	.5898	-.0778	-.668
Germany	315.6	6	35	.0797	.5736	.494
Israel	457.5	4	23	-.4092	-.4861	-.077
Italy	283.4	3	21	-1.0361	-.7679	.268
Japan	175.3	3	8	-1.0361	-1.7770	-.741
Mexico	269.7	3	23	-1.0361	-.4861	.550
Portugal	237.7	4	25	-.4092	-.2352	.174
Switzerland	331.3	7	33	.5898	.3169	-.273
United Kingdom	454.8	6	38	.0797	.8822	.802
United States	774.2	8	39	1.3781	1.5349	.157

Source: Wolfe [11]

After making these adjustments we obtain values of -.183 for T_{1z}, and -.294 for the Spearman rank correlation coefficient. Both of these values indicate that X_1 is more strongly correlated with X_2 than with X_3, which is consistent with the values mentioned in Section 2.

4. Alternate Approaches and a Related Problem

The test procedures discussed in this note are formulated to operate reliably in the presence of unequal variances. A second but less attractive alternative procedure is to perform a preliminary test of H_o: $\sigma_2^2 = \sigma_3^2$ and use Wolfe's test if that hypothesis is not rejected. Noting that X_2 and X_3 are not independent, and that standard nonparametric methods for the paired-sample scale problem appear not to be well known, a simple method is given here. For bivariate normal pairs the solution is obtainable from a result due to Pitman (see Kendall and Stuart [8] pp.139 and 531).

Let (X, Y) be a bivariate random variable with finite second moments. It is easily shown that the sum and difference are uncorrelated if and only if the two variances are equal. Hence, letting U = X + Y and V = X - Y and noting that Cov(U, V) = $\sigma_x^2 - \sigma_y^2$ it may be seen that a significant indication of positive (negative) correlation between U and V implies that σ_x^2 is greater (less) than σ_y^2. Therefore a test of H_o: $\sigma_x^2 = \sigma_y^2$ can be based upon a rank correlation statistic. For the heart disease data in Table 1 define $U_i = X_{2i} + X_{3i}$ and $V_i = X_{3i} - X_{2i}$ (i = 1, ..., 16). The Spearman rank correlation coefficient for the bivariate sample $\{(U_i, V_i)\}$ is

.968, a highly significant indication that $\sigma_3^2 > \sigma_2^2$. Thus Wolfe's test is not appropriate and a modified procedure such as that proposed in Section 3 is indicated.

Still another approach would be to standardize the X_2 and X_3 values, dividing by their individual sample standard deviations. A case can be made for the legitimacy of treating the differences

$$Z_i'' = \frac{X_{3i}}{S_3} - \frac{X_{2i}}{S_2} \quad , \quad i = 1, \ldots, n$$

in the same fashion that Wolfe treats the Z_i by appealing to a multivariate extension of the Theorem of Fligner, Hogg and Killeen [5] to establish the exchangeability of the Z_i''. Proceeding formally with this approach yields rank correlations between the X_{1i} and Z_i'' that are in close agreement with the results obtained using normal scores, namely, -.244 for Spearman's and -.150 for Kendall's.

Next the asymptotically distribution-free test proposed by Davis and Quade [1] may also be adapted to this problem. This approach relies upon the large sample normality of the U-statistic, which is simply the difference of the two Kendall rank correlation coefficients $T_{12} - T_{13}$. The observed value of this difference is .158 with an estimated standard deviation of .121 and hence is significant ($p < .10$) in the direction opposite of that implied by Wolfe's test.

Finally, the jackknife method should be mentioned as a second asymptotically robust technique. The results of Duncan and Layard [2] suggest that a highly satisfactory approach would be to jackknife the difference of Fisher's Z transformation applied to each of r_{12} and r_{13}. The n pseudovalues arise from omitting each of the trivariate cases one at a time. For a modification to the degrees of freedom of the approximate Student t distribution see Hinkley [6].

5. Summary

The method recently proposed by Wolfe [11] is modified in this note to perform in the presence of unequal variances. In considering the equal variance assumption a simple rank correlation test is proposed for the paired-sample scale problem. While there are a variety of procedures for comparing two related correlation coefficients, additional work is needed to extend them to the case of several coefficients. To assess the relative efficiencies of the various available methods under a variety of realistic joint distributions a Monte Carlo study is probably required.

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