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The following two equations should replace those presented in the text:

$$F_2(q_1, q_2) = \frac{2^{-\frac{m}{2}}}{(1-\rho^2)^{\frac{1}{2}}} \frac{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{2^{-(2N_1+2N_2+3)} \Gamma(N_2+1)}{\Gamma(j+1)\Gamma(k+1)\Gamma(N_1+N_2+2)} \frac{(1-\rho^2)^{\frac{1}{2}}}{2^{j+k}} e^{-\left\{\frac{q_1^2+q_2^2}{2(1-\rho^2)}\right\}} \quad (3.2)$$

where $N_1 = \frac{m}{2} + j - 1$ and $N_2 = \frac{n}{2} + k - 1$.

$$F_3(q_1, q_2) = \frac{2^{-\frac{m}{2}}}{(1-\rho^2)^{\frac{1}{2}}} \frac{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{2^{-(2N_1+N_2+N_3-4)} \Gamma(j+1)\Gamma(k+1)\Gamma(\ell+1)}{2} e^{-\left\{\frac{q_1^2+q_2^2}{2(1-\rho^2)}\right\}} \quad (3.3)$$

where $N_1 = \frac{m}{2} + j - 1$, $N_2 = \frac{n}{2} + k - 1$, and $N_3 = \frac{p}{2} + \ell - 1$.

SIMULTANEOUS TESTING AND ESTIMATION WITH
DEPENDENT CHI-SQUARE RANDOM VARIABLES

by

Richard F. Gunst

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DEPARTMENT OF STATISTICS
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functions are developed concisely utilizing canonical analysis and are strong theoretical justification. In addition, expressions for density Chi-square density function, the approximations derived herein have While some approximations are used in dealing with the bivariate

applied to testing and estimation problems faced by the experimenter. of the bivariate Chi-square distribution which can be immediately The purpose of this study is to develop a theoretical structure

tions.

tions, do not immediately lend themselves to numerical computational functions which, although readily adaptable to theoretical derivations, Many of the results that are obtained, moreover, employ mathematical functions which, although readily adaptable to theoretical derivations, are forced to concentrate attention on restrictive types of dependence. later chapters several authors tackle this problem, but most of them of deriving theoretical results on dependent variables. As shown in such method is currently in use is a consequence of the difficulty interpret or make decisions based on experimental data. That no Chi-square random variables is very apparent to anyone who must The need for a practical method of dealing with dependent

INTRODUCTION

CHAPTER I

in a form which affords straightforward computer programming. Chapters II and III of this paper contain the theoretical derivations of moments and forms of the bivariate Chi-square density function. As a result of the study of the moments an approximation is suggested so that a relatively simple form of the density function can be used regardless of the type of dependence. Chapter IV is concerned with simultaneous hypothesis testing problems for which the bivariate Chi-square distribution is a valuable aid. Various procedures for simultaneous variance component estimation are presented in Chapter V. These procedures enable one to validly handle a number of estimation problems which, heretofore, could not be solved. Finally, Chapter VI deals with the determination of critical points for the bivariate Chi-square distribution and the bivariate F distribution. The bivariate Chi-square random variable is related to the bivariate F random variable by means of a simple transformation so attention need not be focused on the bivariate F distribution. Some critical points for these two distributions are presented in Appendix I.

distribution of two quadratic forms of normal random variables are

In this chapter the χ^2 and moments of the bivariate χ^2 -square

taining to the bivariate χ^2 -square distribution.

recently, Jensen (1970) provides a good discussion of many aspects per-

but concentrate on the bivariate χ^2 -square distribution. More

Kibble (1941) and Soltani (1959) discuss much of the same material,

bution and that of the accompanying multivariate normal distribution.

between the correlation matrix of the multivariate χ^2 -square distri-

of the multivariate χ^2 -square distribution and shows the relationship

multivariate Gamma distribution. Krishnaiah (1963) presents the χ^2

tion the authors deduce the corresponding density function for the

of the multivariate χ^2 -square distribution. Using this density func-

distribution, include a derivation of the mgf and the density function

(1951), writing on the more general topic of the multivariate Gamma

discussed by a number of authors. Krishnamoorthy and Parthasarathy

function (χ^2) of the multivariate χ^2 -square distribution have been

The moment generating function (mgf) and the characteristic

1. Introduction

MOMENTS OF THE BIVARIATE χ^2 -SQUARE DISTRIBUTION

CHAPTER II

derived for two cases: the equicorrelated case and the general case. First, canonical analysis is used on two quadratic forms to show that only canonical variables need be discussed throughout the remainder of this paper. Using canonical variables the chf of the equicorrelated case and the chf of the general case are derived. The moments of the two cases are then compared and an important approximation to the general case using the more tractable equicorrelated case is discussed.

2. Canonical Analysis

Let $\bar{Y} \sim N(\bar{0}, \sigma^2 I)$ be an n -variate normal random vector. If $\bar{Y}' A \bar{Y}$ and $\bar{Y}' B \bar{Y}$ are two positive semi-definite quadratic forms with $A(n \times n)$ and $B(n \times n)$ both idempotent, then

$$\bar{Y}' A \bar{Y} / \sigma^2 \sim \chi^2(a) \quad \text{and} \quad \bar{Y}' B \bar{Y} / \sigma^2 \sim \chi^2(b)$$

where

$$a = \text{rank}(A) \leq n$$

$$b = \text{rank}(B) \leq n$$

and $\chi^2(p)$ is a central Chi-square random variable with p degrees

of freedom.

A necessary and sufficient condition (see e.g. Graybill (1961)) for these two random variables to be independent is that

$$AB = \Phi$$

where Φ is an $n \times n$ matrix of zeros. In this case the joint density of these two random variables is simply the product of the marginal

densities. Also, the joint characteristic function is the product of the individual characteristic functions.

A situation frequently encountered in experimentation, including many designed experiments, is that $AB \neq \Phi$ and, therefore, $\bar{Y}'A\bar{Y}/\sigma^2$ and $\bar{Y}'B\bar{Y}/\sigma^2$ are not independent. A useful procedure when faced with this predicament is first to reduce the problem by means of canonical analysis. This technique is presented here and then in the sequel only canonical variables are discussed.

Since A and B are idempotent there exist orthogonal matrices

E ($n \times n$) and F ($n \times n$) such that

$$E'AE = D_1 = \begin{bmatrix} I & \Phi \\ \Phi & \Phi \end{bmatrix} \quad \text{and} \quad F'BF = D_2 = \begin{bmatrix} I & \Phi \\ \Phi & \Phi \end{bmatrix}$$

where D_1 has the first a diagonal elements equal to unity and all other elements are zero. Similarly, D_2 has the first b diagonal elements

equal to unity and all other elements are zero. Let

$$U' = [I \ \Phi] D_1 E' \quad \text{where} \quad I (a \times a), \ \Phi (a \times n - a)$$

$$V' = [I \ \Phi] D_2 F' \quad \text{where} \quad I (b \times b), \ \Phi (b \times n - b)$$

and make the transformations

$$\bar{Z} = 1/\sigma U' \bar{Y} \sim N(\bar{0}, U'U)$$

$$\text{and} \quad \bar{W} = 1/\sigma V' \bar{Y} \sim N(\bar{0}, V'V).$$

The transformed vectors have the joint distribution

$$\begin{pmatrix} \bar{Z} \\ \bar{W} \end{pmatrix} = 1/\sigma \begin{bmatrix} U' \\ V' \end{bmatrix} \bar{Y} \sim N(\bar{0}, \Sigma);$$

but,

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} U'U & U'V \\ V'U & V'V \end{bmatrix} = \begin{bmatrix} I & U'V \\ V'U & I \end{bmatrix} \quad (2.1)$$

Note that the entire relationship between the two random variables

$$\bar{y}'A\bar{y}/\sigma^2 = \bar{z}'\bar{z} \text{ and } \bar{y}'B\bar{y}/\sigma^2 = \bar{w}'\bar{w} \text{ is contained in the covariance}$$

matrix $U'V$. In order to discern more clearly the nature of this relationship a transformation from the vectors $\bar{z}$ and $\bar{w}$ to the canonical vectors

$\bar{r}$, $\bar{s}$, and $\bar{t}$ is now made.

With no loss of generality assume $a \leq b$. Then (see e.g. Anderson

(1958)) there exist matrices $L(a \times a)$, $M(b \times a)$, and $N(b \times b - a)$ such that

$$L'L = I(a \times a) \quad \Sigma_{12}N = \Phi(a \times b - a)$$

$$M'M = I(a \times a) \quad L'\Sigma_{12}M = P = \text{diag}(\rho_1, \rho_2, \dots, \rho_a)$$

$$N'N = I(b - a \times b - a)$$

Where $\rho_1, \rho_2, \dots, \rho_a$ are the non-negative canonical correlations be-

tween the vectors $\bar{r}$ and $\bar{s}$. Also,

$$\begin{pmatrix} \bar{r} \\ \bar{s} \\ \bar{t} \end{pmatrix} = \begin{bmatrix} L' & \Phi \\ \Phi & M' \\ N' & \Phi \end{bmatrix} \begin{pmatrix} \bar{z} \\ \bar{w} \end{pmatrix} \sim N \left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} I & P \\ \Phi & I \\ \Phi & I \end{pmatrix} \right] \quad (2.2)$$

The squared canonical correlations are found from the determinantal

equation

$$\left| \rho^2 \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \right| = 0 \quad (2.3)$$

and the number of non-zero solutions ($\leq a$) to this equation is the

number of correlated variables between the vectors $\bar{r}$ and $\bar{s}$. Note that

these are pairwise dependencies, i.e. each variable in $\bar{r}$ can be

correlated with at most one variable in $\bar{s}$, and there are no other

dependencies between any other variables in $\bar{r}$, $\bar{s}$, and $\bar{t}$.

To solve equation (2.3) one notes that (by eq. (2.1)) this

equation reduces to

$$(2.4) \quad \left| \rho^2 I - U'VV'U \right| = 0.$$

Substituting the appropriate matrices for U and V (2.4) becomes

$$(2.5) \quad \left| \rho^2 I - [I \Phi] E'BE \begin{bmatrix} I \\ \Phi \end{bmatrix} \right| = 0.$$

Recall now that the non-zero latent roots of the product of two matrices

are the same as the non-zero latent roots of the product of the two

matrices multiplied in reverse order, provided the matrices are of the

proper dimensions. So the non-zero solutions to (2.5) are identical to

those of

$$\left| \rho^2 I - E \begin{bmatrix} I \\ \Phi \end{bmatrix} [I \Phi] E'B \right| = 0$$

or,

$$(2.6) \quad \left| \rho^2 I - AB \right| = 0.$$

There will be $k \leq a$ positive solutions to (2.6) and the positive square

roots of these solutions are the k non-zero diagonal elements of P.

Thus, the canonical correlations are the latent roots of AB, where A

and B are the matrices of the original quadratic forms.

Summarizing, the two quadratic forms $\bar{y}'A\bar{y}/\sigma^2$ and $\bar{y}'B\bar{y}/\sigma^2$

are transformed to three quadratic forms such that

$$\bar{y}'A\bar{y}/\sigma^2 = \bar{r}'\bar{r} \quad \text{and} \quad \bar{y}'B\bar{y}/\sigma^2 = \bar{s}'\bar{s} + \bar{t}'\bar{t}$$

and the joint density of $\bar{r}$, $\bar{s}$, and $\bar{t}$ is given by (2.2). The only

dependencies between these canonical variables is contained in k

pairwise correlations between elements of $\bar{r}$ and $\bar{s}$, and the squared correlations are found from (2.6).

3. Characteristic Functions

Suppose one wishes to determine the joint chf of

$$q_1 = \bar{Y}'A\bar{Y}/\sigma^2 = \bar{r}'\bar{r} \quad \text{and} \quad q_2 = \bar{Y}'B\bar{Y}/\sigma^2 = \bar{s}'\bar{s} + \bar{t}'\bar{t}.$$

If no $\rho_i = 1$, then this is

$$\Phi(u_1, u_2) = E \left[e^{iu_1 q_1 + iu_2 q_2} \right]$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{iu_1(\bar{r}'\bar{r}) + iu_2(\bar{s}'\bar{s} + \bar{t}'\bar{t})}$$

$$* \frac{z}{a+b} (2\pi)^{-} \left| \Sigma \right|^{-1/2} e^{-1/2 \bar{h}^{-1} \Sigma^{-1} \bar{h}} d\bar{h}$$

$$\text{where } \bar{h} = \begin{pmatrix} \bar{r} \\ \bar{s} \\ \bar{t} \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} I & P & \Phi \\ \Phi & I & \Phi \\ I & \Phi & I \end{bmatrix}$$

$$\text{Let } C = \begin{bmatrix} iu_1 I & \Phi & \Phi \\ \Phi & iu_2 I & \Phi \\ \Phi & \Phi & iu_2 I \end{bmatrix}$$

where the first two identity matrices are $(a \times a)$ and the third is

($b - a \times b - a$). Then

$$\Phi(u_1, u_2) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{z}{a+b} (2\pi)^{-} \left| \Sigma \right|^{-1/2}$$

$$* e^{-1/2 \bar{h}^{-1} \Sigma^{-1} \bar{h}} (2\pi)^{-} \bar{h} d\bar{h}$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left| \Sigma \right|^{-1/2} \left| \Sigma^{-1} - 2C \right|^{-1/2} \frac{a+b}{2} \left| \Sigma^{-1} - 2C \right|^{1/2} * (2\pi) \frac{a+b}{2} \left| \Sigma^{-1} - 2C \right|^{1/2} e^{-1/2 \bar{h}^T (\Sigma^{-1} - 2C) \bar{h}} d\bar{h}$$

$$= \left| \Sigma \right|^{-1/2} \left| \Sigma^{-1} - 2C \right|^{-1/2}$$

$$= \begin{vmatrix} (1 - 2^{-1} u_1^T) I & -2^{-1} u_1^T P & \Phi \\ -2^{-1} u_2^T P & (1 - 2^{-1} u_2^T) I & \Phi \\ \Phi & \Phi & (1 - 2^{-1} u_2^T) I \end{vmatrix}^{-1/2}$$

$$= \frac{a-k}{2} (1 - 2^{-1} u_1^T) \frac{2}{b-k} - \frac{2}{\pi} \quad j=1$$

$$* \left[(1 - 2^{-1} u_1^T) (1 - 2^{-1} u_2^T) + 4 u_1^T u_2 \right]^{-1/2} \quad (2.7)$$

If any of the $\rho_j = 1$, Σ is singular and the above derivation cannot be

used. The result is, however, the same. Suppose, for example,

$\rho_1 = 1$, then

$$\Phi(u_1, u_2) = E \left[e^{i u_1^T \bar{x} + i u_2^T (\bar{s}^T + i t)} \right]$$

$$\Phi(u_1, u_2) = \left[(1 - 2iu_1)(1 - 2iu_2) + 4u_1u_2 \right]^{-1/2} \left[(1 - 2iu_1)(1 - 2iu_2) + 4u_1u_2 \right]^{-1/2} \frac{2}{b-k} \pi \quad j=2$$

$$\begin{aligned} \left[\begin{array}{c} z_n I_{n \times 1} + (z_n I_{n \times 1} - I)(I_{n \times 1} z_n - I) \\ z_n I_{n \times 1} + (z_n I_{n \times 1} - I)(I_{n \times 1} z_n - I) \end{array} \right] &= \\ \left[\begin{array}{c} z_n I_{n \times 1} + (z_n I_{n \times 1} - I)(I_{n \times 1} z_n - I) \\ z_n I_{n \times 1} + (z_n I_{n \times 1} - I)(I_{n \times 1} z_n - I) \end{array} \right] &= \\ \left[\begin{array}{c} z_n I_{n \times 1} + (z_n I_{n \times 1} - I)(I_{n \times 1} z_n - I) \\ z_n I_{n \times 1} + (z_n I_{n \times 1} - I)(I_{n \times 1} z_n - I) \end{array} \right] &= \end{aligned}$$

But since the correlation between r_1 and s_1 is unity,

$$* \left[(1 - 2i u_1)(1 - 2i u_2) + 4u_1 u_2 \rho_j^2 - 1/2 \right]$$

$$\Phi(n, l, z_n) = \left[e^{l_n l_{\pi} l_{\alpha} l_{\beta}} + z_n^{l_{\beta}} z_n^{l_{\alpha}} \right] \frac{z}{a - \frac{z}{k}} - \frac{z}{b - \frac{z}{k}} (1 - z_n^{l_{\beta}}) * \quad j = 2$$

from (2.7),

since r_1 is correlated with s_1 but both are independent of the remaining variables. Also, $r_1^* = (r_2, r_3, \dots, r_a)$ and $\bar{s}^* = (s_2, s_3, \dots, s_a)$. So

$$E = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \quad E^* = \begin{bmatrix} e_{11}^* & e_{12}^* \\ e_{21}^* & e_{22}^* \end{bmatrix}$$

which is identical to (2.7). Hence, eq. (2.7) represents the chf of the bivariate Chi-square distribution regardless of the values of the canonical correlations ρ_j .

If all the canonical correlations are equal, (2.7) reduces to

$$\Phi^I(u_1, u_2) = (1 - 2i u_1)^{-\frac{a-k}{2}} (1 - 2i u_2)^{-\frac{b-k}{2}} \left[(1 - 2i u_1)^{-\frac{2}{k}} - \frac{2}{k} \right] * (1 - 2i u_2)^{-\frac{2}{k}} + 4u_1 u_2 \rho^2 \quad (2.8)$$

This is referred to as the equicorrelated case. If $a = b = k$, this is the same chf as that of the sums of squares of a random sample of size k from a standard bivariate normal population.

4. Moments of the Bivariate Chi-square Distribution

The aim of studying the moments of the bivariate Chi-square distribution is to determine if some function of the individual correlations in the general case can be used to approximate the single

correlation ρ in the equicorrelated case. If a good approximation is found, the need to study the bivariate Chi-square density in the general case will be eliminated. In the following, assume for simplicity that

$$k = a.$$

Consider first the chf (2.8) of the bivariate Chi-square distribution in the equicorrelated case. One need only consider joint

moments since the moments of the marginal random variables do not contain the parameter ρ and must reduce to those of a univariate

Chi-square random variable. The first mixed moment is derived below and other mixed moments are presented without derivation.

Let

$$\mu_{12}^{(j)(k)} = E \left[q_{j+k}^1 q_{j+k}^2 \right] = (-i)^{j+k} \frac{\partial^{j+k} \Phi_1(u_1, u_2)}{\partial u_1^j \partial u_2^k} \bigg|_{u_1 = u_2 = 0}.$$

Then,

$$\mu_{12}^{(1)(1)} = (-i)^2 \frac{\partial^2 \Phi_1(u_1, u_2)}{\partial u_1 \partial u_2} \bigg|_{u_1 = u_2 = 0}.$$

$$= (-i) \frac{\partial}{\partial u_2} \left[(1 - 2iu_2) \frac{z}{b - a} \left(-\frac{z}{a} \right) \right]$$

$$* \left[(1 - 2iu_1)(1 - 2iu_2) + 4u_1u_2 \right] \frac{z}{a + 2}$$

$$* \left\{ -2i(1 - 2iu_2) + 4u_2 \right\} \frac{z}{a + 2} \bigg|_{u_1 = u_2 = 0}$$

$$= (-i) \left[\left(-\frac{z}{b - a} \right) (1 - 2iu_2) \frac{z}{a + 2} - (-2i) \left(-\frac{z}{a} \right) \right]$$

[illegible]

$$\begin{aligned}
& * \sum_{a=1}^j (-1/2) \left[(1 - 2 \pm u_1) (1 - 2 \pm u_2) + 4 u_1 u_2 \rho_j^k \right]^{2 - 3/2} \\
& + \left[-\frac{2}{b-a} + (1 - 2 \pm u_2) \right]^{2 - 1/2} \\
& + \sum_{a=1}^{\pi} \left\{ \sum_{k=1}^{\pi} \rho_j^k (1 - 2 \pm u_1) (1 - 2 \pm u_2) + 4 u_1 u_2 \right\}^{2 - 5/2} \\
& * \left\{ \sum_{k=1}^{\pi} \rho_j^k (1 - 2 \pm u_1) (1 - 2 \pm u_2) + 4 u_1 u_2 \right\}^{2 - 3/2} \\
& * \left[(1 - 2 \pm u_1) (1 - 2 \pm u_2) + 4 u_1 u_2 \rho_j^k \right]^{2 - 5/2} \\
& - \frac{2}{b-a} \sum_{a=1}^j (-1/2) (-3/2) \\
& * \sum_{a=1}^{\pi} \left[(1 - 2 \pm u_1) (1 - 2 \pm u_2) + 4 u_1 u_2 \rho_j^k \right]^{2 - 1/2} \\
& * \left\{ \sum_{k=1}^{\pi} \rho_j^k (1 - 2 \pm u_1) (1 - 2 \pm u_2) + 4 u_1 u_2 \right\}^{2 - 1/2} \\
& * \left[(1 - 2 \pm u_1) (1 - 2 \pm u_2) + 4 u_1 u_2 \rho_j^k \right]^{2 - 1/2}
\end{aligned}$$

with the above substitution. But even in $\mu_{12}^{(2)(2)}$, the difference in the

both sets of mixed moments through the fourth degree are identical

$\underline{p.2}$ in the general case. So except for the last term of $\mu_{12}^{(2)(2)}$,

identical in both cases if $\underline{p.2}$ in the equicorrelated case is replaced by

Note in this table that all mixed moments except the last one are

2.1.

substitution. Other moments, derived as above, are found in Table

bivariate Chi-square distribution in the equicorrelated case with this

Chi-square distribution in the general case can be made using the

indicates that a good approximation to the joint density of the bivariate

equation is identical to eq. (2.10). A comparison of other moments

and (2.10). If one replaces $\underline{p.2}$ in eq. (2.9) with $\underline{p.2}$, the resulting

One immediately notes the similarity between equations (2.9)

$$\text{where } p.2 = \sum_{j=1}^a p_j^2 \text{ and } \underline{p.2} = \frac{1}{a} p.2.$$

(2.10)

$$= ab + 2a p.2 = ab + 2a p.2 + (b-a)a - 2a + 2 p.2 + 3a + a(a-1)$$

$$\left[\sum_{j=1}^a \sum_{i=1}^3 (i) + \sum_{j=1}^a \sum_{k=1}^3 (i) \right] \quad k \neq j$$

$$= (-1) \left[(b-a) \sum_{j=1}^a (i) + \sum_{j=1}^a (2 - 2 p_j^2) + \right]$$

moments is numerically quite small since $-1 \leq \rho_j \leq 1$ and, therefore, each ρ_j^4 will be small. The first terms in this moment should heavily outweigh the last ones. Hence, the first four mixed moments are essentially identical with ρ^2 replaced by ρ_j^2 and so the bivariate Chi-square density in the equicorrelated case with this substitution should be an excellent approximation to the bivariate Chi-square density in the general case.

TABLE 2.1

Comparison of Moments of the Bivariate Chi-square
Distribution in the Equicorrelated and General Cases

Moment	Equicorrelated Case	General Case
$\mu_{12}^{(1)(1)}$	$ab + 2ap$	$ab + 2ap$
$\mu_{12}^{(1)(2)}$	$ab^2 + 2ab + 4abp + 8ap^2$	$ab^2 + 2ab + 4abp + 8ap^2$
$\mu_{12}^{(2)(1)}$	$a^2b + 2ab + 8ap^2 + 4a^2p$	$a^2b + 2ab + 8ap^2 + 4a^2p$
$\mu_{12}^{(1)(3)}$	$ab^3 + 6ab^2 + 8ab + 6abp$	$ab^3 + 6ab^2 + 8ab + 6abp$
$\mu_{12}^{(3)(1)}$	$a^3b + 6a^2b + 8ab + 6a^3p$	$a^3b + 6a^2b + 8ab + 6a^3p$
$\mu_{12}^{(2)(2)}$	$a^2b^2 + 2ab^2 + 2a^2b + 4ab$	$a^2b^2 + 2ab^2 + 2a^2b + 4ab$
	$+ 8a^2bp + 16abp^2$	$+ 8a^2bp + 16abp^2$
	$+ 16a^2p^2 + 32ap^3$	$+ 16a^2p^2 + 32ap^3$
	$+ 8a^2p^4 + 16ap^4$	$+ 8a^2p^4 + 16ap^4$

CHAPTER III

THE BIVARIATE CHI-SQUARE DENSITY FUNCTION

1. Introduction

Discussion of the bivariate Chi-square and bivariate F density

functions are of a somewhat limited scope in the literature. Kibble

(1941), Soitani (1959), and Krishniah (1963) derive the bivariate Chi-

square density for the equicorrelated case, i. e. where all the canonical

correlations of the underlying normal canonical variables are equal,

and require that each of the marginal Chi-square distributions have

identical degrees of freedom. Jensen (1970) presents the density function

in its most general form and discusses some theoretical properties of

the bivariate Chi-square distribution. The major drawback to Jensen's

discussion is that his formulation requires the use of Laguerre polynomials

and convoluted summations--ideal for theoretical derivations but difficult

to use in numerical computations.

The bivariate F distribution likewise suffers from a limited

discussion in the literature. Ramachandran (1956) proposes a method

for computing probabilities using k F statistics with independent Chi-

square random variables in the numerators but the same independent

Chi-square random variable in the denominators. He makes no

restrictions on the degrees of freedom of the numerator Chi-square

random variables, yet he also shows the simplification of computational procedures if the degrees of freedom are identical. Extensive tabula-

tion of upper and lower critical points for the latter case has been done

by Armitage and Krishnaiah (1964a, 1964b). Krishnaiah (1963),

Krishnaiah and Armitage (1965), and Jensen (1970) examine the bivariate

F distribution with dependent numerator Chi-square random variables

and the same independent Chi-square random variable in all the

denominators.

In this chapter the density function of the bivariate Chi-square

distribution is discussed. Derivations are restricted to the equicorrelated

case since an excellent approximation to the density function in the general

case can be made by using the equicorrelated density and replacing ρ^2

by ρ^2 . Three forms of the density function are derived corresponding

to different types of dependence and different degrees of freedom in the

marginal Chi-square random variables.

2. Density Functions

For notational purposes let $(q_1, q_2) \sim \text{Biv}\chi^2(a, b, c)$ denote that

the random variables q_1 and q_2 have a bivariate Chi-square distribution,

where

a = degrees of freedom of the marginal Chi-square distribution

of q_1 ,

b = degrees of freedom of the marginal Chi-square distribution

of q_2 ,

c = number of non-zero canonical correlations between the

underlying normal vectors of q_1 and q_2 .

The first case to be considered is the one considered by Kibble

(1941), Soitani (1959), and Krishnaiah (1963). Let

$$q_1 \sim \chi^2(m) \quad \text{and} \quad q_2 \sim \chi^2(m) \quad \text{and} \quad \bar{v}'\bar{v} = \bar{v}'\bar{v} \sim \chi^2(m)$$

where

$$\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \sim N \left[\begin{pmatrix} \bar{0} \\ \bar{0} \end{pmatrix}, \begin{pmatrix} I & P \\ P & I \end{pmatrix} \right]$$

with $P(m \times m) = P I$ and $P > 0$. Then

$$(q_1, q_2) \sim \text{Biv} \chi^2(m, m, m).$$

The joint density of q_1 and q_2 is

$$f(q_1, q_2) = \frac{(1 - \rho^2)^{-\frac{m}{2}}}{\sum_{j=0}^{\infty} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2} + j) \Gamma(j+1)}} \quad j=0$$

$$* \left[\frac{\rho^{2(1-\rho^2)}}{2^j} \right] N_1(q_1, q_2) \exp \left\{ - \frac{q_1 + q_2}{2(1-\rho^2)} \right\} \quad (3.1)$$

where $N_1 = \frac{m}{2} + j - 1$. Equation (3.1) is identical to the form first

derived by Soitani, and is absolutely convergent (see Jensen (1970) for

$$\rho > 1.$$

Now suppose one encounters the situation where

$$q_1 \sim \chi^2(m) \quad \text{and} \quad q_2 \sim \chi^2(m+n), \quad n > 0$$

and there are m non-zero equal canonical correlations between the two

sets of normal vectors making up q_1 and q_2 . One is, then, concerned

with finding the joint density of q_1 and q_2 , where $(q_1 q_2) \sim \text{Biv} \chi^2(m, m+n, m)$

with parameter ρ . To find this density begin with two independent

densities $(s_1 s_2) \sim \text{Biv} \chi^2(m, m, m)$ and

$$(t_1, t_2) \sim \text{Biv} \chi^2(n, n, n)$$

each with the same parameter ρ . Make the transformations

$$q_1 = s_1, \quad q_2 = s_2 + t_2 \sim \chi^2(m+n)$$

$$q_3 = t_1$$

$$q_4 = s_2 - t_2.$$

Integrating out q_3 and q_4 one obtains

$$\int_0^\infty \int_{q_2}^{q_2 - q_4} f(q_1, q_2) = \int_{q_2}^{q_2 - q_4} f(q_1, q_2, q_3, q_4) dq_3 dq_4$$

$$= \int_0^\infty \int_{q_2}^{q_2 - q_4} \frac{\Gamma(\frac{m}{2}) \Gamma(\frac{n}{2})}{\Gamma(\frac{m+n}{2})} (1 - \rho^2)^{\frac{m+n}{2} - 1} dq_2 dq_4$$

$$* \sum_{k=0}^{\infty} \frac{\Gamma(\frac{m}{2}) \Gamma(\frac{n}{2}) \Gamma(k+1) \Gamma(k+1)}{\Gamma(\frac{m}{2}) \Gamma(\frac{n}{2}) \Gamma(k+1) \Gamma(k+1)} = 1$$

$$* \left[\frac{1}{z^p} \right]_{2j+2k} \frac{z}{m} + j - 1 \exp \left\{ - \frac{q_1 + q_2}{z} \right\}$$

$$* \frac{m}{z} + j - 1 (q_2 + q_4) \frac{z}{n} + k - 1 (q_2 + q_4)$$

$$* \frac{z}{n} + k - 1 \exp \left\{ - \frac{q_3}{z} \right\} \frac{z}{p} q_4 q_3$$

$$= \frac{\sum_{j=0}^{\infty} \frac{(1 - \frac{z}{m})^j}{z} \Gamma(\frac{z}{m}) \Gamma(\frac{z}{n})}{\sum_{k=0}^{\infty} \frac{z}{z - (\frac{3m}{2} + n + 3j + 2k - 2)} \Gamma(\frac{z}{m} + j) \Gamma(j + 1) \Gamma(k + 1)}$$

$$* \frac{p}{2j + 2k} \frac{(1 - \frac{z}{p})^{2j+k}}{z} q_1 q_2 N_1 N_2 + 1$$

$$* \exp \left\{ - \frac{q_1 + q_2}{z} \right\} \sum_{j=0}^{\infty} [N_1] \sum_{k=0}^{\infty} [N_2] \begin{pmatrix} N_1 \\ q_1 \end{pmatrix} \begin{pmatrix} N_2 \\ q_2 \end{pmatrix}$$

$$* (-1)^{r_2} \frac{\delta(r_1 + r_2 + 1)}{(r_1 + r_2 + 1)} \quad (3.2)$$

where $N_1 = \frac{m}{2} + j - 1, N_2 = \frac{n}{2} + k - 1, [N_1] =$ greatest integer $\leq N_1$,
 and $\delta(x) = \begin{cases} 1 & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even} \end{cases}$

The final case which must be considered occurs when

$$q_1 \sim \chi^2_{m+n}, n > 0 \text{ and } q_2 \sim \chi^2_{m+p}, p > 0$$

but there are only m non-zero equal canonical correlations. Then,

$$(q_1, q_2) \sim \text{Biv} \chi^2_{m+n, m+p, m}.$$

This situation is handled similarly to the above one. Begin with three

independent bivariate Chi-square random variables:

$$(s_1, s_2) \sim \text{Biv} \chi^2_{m, m, m}$$

$$(t_1, t_2) \sim \text{Biv} \chi^2_{n, n, n}$$

and

$$(u_1, u_2) \sim \text{Biv} \chi^2_{p, p, p}$$

all with the same parameter p . Make the transformations

$$q_1 = s_1 + t_1 \sim \chi^2_{m+n}$$

$$q_2 = s_2 + u_2 \sim \chi^2_{m+p}$$

$$q_3 = s_1 - t_1$$

$$q_4 = s_2 - u_2$$

$$q_5 = t_2$$

$$q_6 = u_1.$$

and integrate out $q_3 (-q_1 < q_3 < q_1)$, $q_4 (-q_2 < q_4 < q_2)$, $q_5 (q_5 > 0)$, and $q_6 (q_6 > 0)$. This results in the following density function:

derived if one is only concerned with F statistics that have a common
The density function for the Bivariate F Distribution need not be

3. Density Function of the Bivariate F Distribution

where $N_1 = N_2 = \frac{m}{2} + j - 1$, $N_3 = \frac{n}{2} + k - 1$, and $N_4 = \frac{p}{2} + \ell - 1$.

$$* \frac{(x_1 + x_3 + 1)(x_2 + x_4 + 1)}{\delta(x_1 + x_3 + 1)\delta(x_2 + x_4 + 1)} \quad (3.3)$$

$$* \begin{pmatrix} N_1 \\ x_1 \end{pmatrix} \begin{pmatrix} N_2 \\ x_2 \end{pmatrix} \begin{pmatrix} N_3 \\ x_3 \end{pmatrix} \begin{pmatrix} N_4 \\ x_4 \end{pmatrix} (-1)^{x_3 + x_4}$$

$$* \exp \left\{ - \frac{q_1 + q_2}{2(1 - p_2)} \right\} \begin{matrix} [N_1] \\ x_1 = 0 \\ \Sigma \end{matrix} \begin{matrix} [N_2] \\ x_2 = 0 \\ \Sigma \end{matrix} \begin{matrix} [N_3] \\ x_3 = 0 \\ \Sigma \end{matrix} \begin{matrix} [N_4] \\ x_4 = 0 \\ \Sigma \end{matrix}$$

$$* \frac{p^{2j + 2k + 2\ell} q_1^{2j + k + \ell} q_2^{2j}}{N_1 + N_3 + 1 \quad N_2 + N_4 + 1}$$

$$* \frac{\Gamma(\frac{m}{2} + j) \Gamma(j + 1) \Gamma(k + 1) \Gamma(\ell + 1)}{2^{-(2m + n + p + 4j + 2k + 2\ell - 4)}}$$

$$f_3(q_1, q_2) = \frac{\Gamma(\frac{m}{2}) \Gamma(\frac{n}{2}) \Gamma(\frac{p}{2})}{2^{-(1 - p_2)} - \frac{m}{2}} \begin{matrix} \Sigma \\ j = 0 \\ \infty \end{matrix} \begin{matrix} \Sigma \\ k = 0 \\ \infty \end{matrix} \begin{matrix} \Sigma \\ \ell = 0 \\ \infty \end{matrix}$$

Chi-square random variable in the denominator. Probabilities and critical points for this case can be computed from the bivariate Chi-square density and an independent univariate Chi-square density as shown in Chapter VI.

CHAPTER IV

SIMULTANEOUS TESTS OF STATISTICAL HYPOTHESES

1. Introduction

The preceding chapters were devoted to developing a theoretical foundation upon which testing and estimation procedures can be built. In this chapter simultaneous testing procedures are discussed and applied. Only two factor designs with no interactions are investigated but extensions to other designs and analyses are straightforward.

Naturally when one designs an experiment he chooses, subject to economic and time requirements, a design which has certain desirable properties. In particular, he likes his design to be

balanced: an equal number of observations occur in each

subclass of the model, and

complete: all treatments occurring in each block.

There are many advantages to designs which are balanced and complete, but perhaps the most important one is that both block and treatment effects are estimated independently of one another. If two independent estimates of random error exist then each effect can also be tested independently. This is not true for most incomplete or unbalanced designs.

Unfortunately economic considerations and time constraints do not always permit one to use complete block designs. More frequently, accidents during experimentation or other physical considerations lead to an imbalance in the design. In these situations currently used analyses are handicapped when investigating more than one of the factors under study. This is a loss to the experimenter in both wasted information and finances. Using the densities derived in the previous chapter and the theory of "Simultaneous Analysis of Variance," techniques are now developed to handle these situations.

2. Simultaneous Analysis of Variance

"Simultaneous Analysis of Variance" is a term used to refer to the procedure first proposed by M. N. Ghosh (1955) for testing two or more hypotheses. Both Ghosh and Ramachandran (1956) discuss testing two or more hypotheses with "quasi-independent" F statistics. Two or more F statistics are called "quasi-independent" if all the numerator Chi-square random variables are distinct and independent and the Chi-square random variable common to all the denominators is also independent of the numerator Chi-squares. This is the standard Analysis of Variance situation. Krishnaiah [1965a] discusses multiple tests with dependent Chi-square random variables in the numerators and the same independent Chi-square in all the denominators. Again, however, he restricts the numerator Chi-squares to having identical degrees of freedom and all squared canonical correlations positive and equal.

As mentioned above the Simultaneous Analysis of Variance (SANOVA) test is concerned with testing two or more hypotheses. The overall significance level of the test is the probability of rejecting one or more of the hypotheses when all of them are true. For example, if one wishes to test both blocks and treatments in a Balanced Incomplete Block Design (BIBD), the hypotheses may be written

$$H^{01} : \text{no block effect}$$

$$\text{and } H^{02} : \text{no treatment effect.}$$

The SANOVA procedure tests

$$H = H^{01} \cap H^{02},$$

i.e. both H^{01} and H^{02} are true, against the alternative hypothesis that

at least one of them is false. The test has a 100% significance level

if

$$\Pr \left\{ F_T \leq F_{T, \alpha}, F_B \leq F_{B, \alpha} \mid H \right\} = 1 - \alpha \quad (4.1)$$

where

$$F_T = \text{MST/MSE} = (\text{SST}/t - 1) / (\text{SSE}/bk - b - t + 1)$$

$$F_B = \text{MSB/MSE} = (\text{SSB}/b - 1) / (\text{SSE}/bk - b - t + 1)$$

and $F_{T, \alpha}$ and $F_{B, \alpha}$ are constants suitably chosen so that (4.1) is

true. The computation of these critical points will be deferred until

Chapter VI.

When using the SANOVA testing procedure H is rejected if

either $F_B > F_{B, \alpha}$ (and H^{01} is rejected) or $F_T > F_{T, \alpha}$ (and H^{02} is

rejected) or both. If neither H^{01} nor H^{02} is rejected then H , the joint

hypothesis, is not rejected.

3. Balanced Incomplete Block Design

Perhaps the best method of illustrating the technique of

simultaneous testing is with an example. Consider a BIBD in which one is interested in testing for both block and treatment effects. A simultaneous test of both effects is easily developed utilizing the theory of

the previous chapters.

A BIBD consists of assigning t treatments to b blocks each of

which contains k plots ($k < t$). Each treatment occurs a total of r times in the layout and at most once in any block. In addition, any two treatments occur together in exactly λ blocks. The blocks and treatments

can be either fixed or random effects and the arithmetic of the analysis will be the same in both cases. This is due to the joint distribution of F_T and F_B being identical under H in either case. The distributions

under the two alternative hypotheses will differ, but this does not affect the testing procedure. In the following example suppose both effects

are random.

The model for this design is

$$y_{ij} = \mu + \beta_i + \tau_j + e_{ij} \quad i = 1, 2, \dots, b; j = 1, 2, \dots, t \quad (4.2)$$

for each (i, j) combination occurring in the layout. In this model, y_{ij} = yield of the plot in the i^{th} block to which the j^{th} treatment is applied

μ = unknown constant representing the overall mean yield

$$\left. \begin{array}{l} \beta_i = \text{random block effect,} \\ \tau_j = \text{random treatment effect,} \\ \epsilon_{ij} = \text{random error effect,} \end{array} \right\} \begin{array}{l} \beta_i \sim \text{NID}(0, \sigma^2) \\ \tau_j \sim \text{NID}(0, \sigma^2) \\ \epsilon_{ij} \sim \text{NID}(0, \sigma^2) \end{array} \text{ independently}$$

In matrix notation (4.2) is

$$Y = \mu \bar{1} + X_B \bar{\beta} + X_T \bar{\tau} + \bar{\epsilon} \quad (4.3)$$

where

$$\begin{aligned} \bar{Y}' &= (Y_{11}, Y_{12}, \dots, Y_{1k}, \dots, Y_{b1}, Y_{b2}, \dots, Y_{bk}) \\ &= (Y_1, Y_2, \dots, Y_k, \dots, Y_{bk-k+1}, Y_{bk-k+2}, \dots, Y_{bk}) \\ \bar{1}' &= (1, 1, \dots, 1, \dots, 1, 1, \dots, 1) \end{aligned}$$

$$\bar{\beta}' = (\beta_1, \beta_2, \dots, \beta_b)$$

$$\bar{\tau}' = (\tau_1, \tau_2, \dots, \tau_t)$$

$$X_B = [x_{uv}] \quad \text{where} \quad x_{uv} = \begin{cases} 1 & \text{if } y_u \text{ occurs in block } v \\ 0 & \text{otherwise} \end{cases}$$

$$X_T = [x_{uv}] \quad \text{where} \quad x_{uv} = \begin{cases} 1 & \text{if } y_u \text{ receives treatment } v \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{\epsilon}' = (\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{1k}, \dots, \epsilon_{bk-k+1}, \epsilon_{bk-k+2}, \dots, \epsilon_{bk})$$

Now make the transformation

$$Z_B = X_B - X_T (X_T' X_T)^{-1} X_T' X_B$$

where A^{-} represents the generalized inverse of the matrix A (see, for example, Rao (1962)). Two properties of Z_B are

$$Z_B' X_T' = 0 \quad \text{and} \quad Z_B' \bar{1} = 0.$$

An ANOVA for testing the block effect can be written as

ANOVA

Source	d.f.	S.S.	E[M.S.]
Mean	1	$SSM = \frac{1}{bk} \overline{Y'11'Y}$	
Treatments	$t - 1$	$SST = \frac{1}{t} \overline{Y'1X'1'Y} - SSM$	
Blocks (adj)	$b - 1$	$SSB(adj) = \overline{Y'1Z'1'Z} - \frac{1}{b} (\overline{Z'1Z'1'Z})^2$	$\sigma^2 + \left[\frac{\lambda t(t-1)}{k(b-t)} + \frac{r(b-1)}{k(b-t)} \right] \sigma^2$
Error	$bk-b-t+1$	$SSE = Y'1Y - SSM - SST$	σ^2
-SSB(adj)			

In this ANOVA the block effect has been adjusted for the treatment effect and, therefore, H_{01} can be tested using $F_B = MSB/MSE$. H_{02} cannot be tested since the block effect and the treatment effect are confounded in SST. An alternate ANOVA can be derived by initially making the

transformation

$$Z^T = X^T - X^T_B (X^T_B X^T_B)^{-1} X^T_B X^T$$

It is easily verified that Z^T has the following properties
 $Z^T X^T_B = 0$ and $Z^T 1 = 0$.

An ANOVA for testing the treatment effect is

ANOVA

Source	d.f.	S.S.	E[M.S.]
Mean	1	$SSM = \frac{1}{bk} \overline{Y'11'Y}$	
Blocks	$b-1$	$SSB = \frac{1}{k} \overline{Y'1X'1'Y} - SSM$	

ANOVA
(Continued)

Source	d.f.	S.S.	E[M.S.]
Treatments	$t - 1$	$SST(adj) = \bar{Y}' Z^T (Z^T Z^T)^{-1} Z^T Y$	$\sigma^2 + \frac{\lambda t}{k} \sigma^2$
Error	$bk - b - t + 1$	$SSE = \bar{Y}' Y - SSM - SSB - SST(adj)$	σ^2

For the same reasons as in the previous ANOVA, one can test H_0 here but not H_{01} . So in each of the above cases one can test for the significance of one of the factors, but is stymied in any attempt to validly test both.

Referring now to the SANOVA testing procedure, define the

following matrices:

$$A = Z^T (Z^T Z^T)^{-1} Z^T \quad \text{and} \quad B = Z^T (Z^T Z^T)^{-1} Z^T$$

Then under H ,

$$\bar{Y}' A Y / \sigma^2 = SST(adj) / \sigma^2 \sim \chi^2_{(t-1)}$$

and

$$\bar{Y}' B Y / \sigma^2 = SSB(adj) / \sigma^2 \sim \chi^2_{(b-1)}.$$

These two random variables are not independent since $AB \neq \Phi$. Hence a simultaneous test of the two hypotheses using the bivariate Chi-square distribution must be made.

Consider now the two quadratic forms

$$\bar{Y}' A Y = \bar{Y}' Z^T (Z^T Z^T)^{-1} Z^T Y$$

and

$$\bar{Y}' B Y = \bar{Y}' Z^T (Z^T Z^T)^{-1} Z^T Y.$$

Since $Z^T Z$ and $Z^T Z_B$ are real symmetric matrices there exist real orthogonal matrices E^* and F^* such that

$$E^* Z^T Z E^{*T} = D_1^* \quad \text{and} \quad F^* Z^T Z_B F^{*T} = D_2^*,$$

where D_1^* and D_2^* are diagonal matrices with the latent roots of

$$Z^T Z \quad \text{and} \quad Z^T Z_B$$

displayed along the respective diagonals. But,

$$Z^T Z = X^T X - \frac{1}{k} X^T X_B X_B^T X$$

$$= r I - \frac{1}{k} [(r - \lambda) I + \lambda \bar{1} \bar{1}^T]$$

$$= \frac{\lambda t}{k} I - \frac{1}{\lambda} \bar{1} \bar{1}^T.$$

The latent roots of $Z^T Z$ are easily verified to be

$$\lambda_1 = 0, \lambda_2 = \lambda_3 = \dots = \lambda_t = \frac{k}{\lambda t}.$$

Similarly,

$$Z^T Z_B = X^T X_B - \frac{1}{k} X^T X_B X_B^T X_B$$

$$= k I - \frac{1}{k} X^T X_B X_B^T X_B.$$

So the latent roots of $Z^T Z_B$ are the solutions λ_i^* to the determinantal

equation

$$\left| k I - \frac{1}{k} X^T X_B X_B^T X_B - \lambda_i^* I \right| = 0$$

or

$$\left| X^T X_B X_B^T X_B + r (\lambda_i^* - k) I \right| = 0$$

Let $N = X^T X_B$, then this equation becomes

$$\left| N^T N + r (\lambda_i^* - k) I \right| = 0.$$

Since $\text{rank}(N'N) = \text{rank}(NN') = t$, $b-t$ solutions to this determinantal equation are zero; therefore, $b-t$ of the latent roots of $Z_B' Z_B$ are $\lambda^* = k$. The non-zero latent roots of $N'N$ are identical to those of NN' , so the non-zero solutions to this determinantal equation are identical to those of

$$\left| NN' + r(\lambda^* - k)I \right| = 0$$

or

$$\left| (r - \lambda)I + \lambda I I' + r(\lambda^* - k)I \right| = 0.$$

Solving this equation one finds the remaining t latent roots of $Z_B' Z_B$

are

$$\lambda_1^* = 0, \lambda_2^* = \lambda_3^* = \dots = \lambda_t^* = \frac{r}{\lambda t}.$$

Hence

$$D_1^* = \text{diag} \left(\frac{\lambda t}{k}, \frac{\lambda t}{k}, \dots, \frac{\lambda t}{k}, 0 \right)$$

and

$$D_2^* = \left(\frac{\lambda t}{r}, \frac{\lambda t}{r}, \dots, \frac{\lambda t}{r}, k, k, \dots, k, 0 \right).$$

The row vectors in E_1^* and F_1^* corresponding to the zero latent roots are, respectively, $1/\sqrt{t}$ and $1/b$ I' . If these vectors are removed from E_1^* and F_1^* and the remaining matrices denoted by E and F , then

$$EZ_1' Z_1' E' = D_1 = \text{diag} \left(\frac{\lambda t}{k}, \frac{\lambda t}{k}, \dots, \frac{\lambda t}{k} \right)$$

and

$$F'Z_B'Z_B'F' = D_2 = \text{diag} \left(\frac{r}{\lambda t}, \frac{r}{\lambda t}, \dots, \frac{r}{\lambda t}, k, k, \dots, k \right) .$$

Now make the transformations

$$\underline{z} = D_1^{-1/2} E' Z_1' Y \quad \text{and} \quad \underline{w} = D_2^{-1/2} F' Z_B' Y$$

so that

$$(4.4) \quad \left\{ \begin{array}{l} \underline{z}' \underline{z} = Y' Z_1' E D_1^{-1} E' Z_1' Y \\ \text{and} \quad \underline{w}' \underline{w} = Y' Z_B' F D_2^{-1} F' Z_B' Y \end{array} \right. .$$

Recall that

$$E' Z_1' Z_1' E' = D_1 ,$$

therefore,

$$E' E' Z_1' Z_1' E' E' = E' D_1 E$$

or,

$$Z_1' Z_1' = E' D_1 E$$

(4.5)

$$\text{since } E' E = I - \frac{1}{t} \underline{1} \underline{1}' \quad \text{and} \quad \underline{1}' Z_1' = \underline{0}' . \quad \text{Thus,}$$

$$(Z_1' Z_1')^{-1} = (E D_1 E)^{-1} = (E' D_1^{-1} E)^{-1} = E' D_1^{-1} E$$

$$= E' D_1^{-1} E$$

$$= E' D_1^{-1} E .$$

Similarly,

$$(Z_B' Z_B')^{-1} = F' D_2^{-1} F ,$$

so eq. (4.4) become

$$(4.6) \quad \left\{ \begin{array}{l} \underline{z}' \underline{z} = Y' Z_1' (Z_1' Z_1')^{-1} Z_1' Y = \text{SST (adj)} \\ \text{and} \quad \underline{w}' \underline{w} = Y' Z_B' (Z_B' Z_B')^{-1} Z_B' Y = \text{SSB (adj)} \end{array} \right.$$

special case $b = t$, and recalling that there can be at most $t - 1$ non-

zero canonical correlations between the $t-1$ variables in $\bar{z}$ and the $b-1$

variables in $\bar{w}$. Thus,

$$\begin{pmatrix} \bar{z} \\ \bar{w} \end{pmatrix} \sim N \begin{pmatrix} \bar{0} \\ \bar{0} \end{pmatrix}, \begin{bmatrix} \sigma_2^2 I + \sigma_2^2 D_1 & -\sqrt{\frac{r_k}{1-\lambda}} \sigma_2^2 I \\ -\sqrt{\frac{r_k}{1-\lambda}} \sigma_2^2 I & \sigma_2^2 I + \sigma_2^2 D_2 \end{bmatrix} \begin{bmatrix} \bar{\Phi} \\ \bar{\Phi} \end{bmatrix} \quad (4.7)$$

So under $H = H_{01} \cup H_{02}$,

$$\left(\frac{1}{2} \text{SSB (adj)}, \frac{1}{2} \text{SST (adj)} \right) \sim \text{Biv } \chi^2 (b-1, t-1, t-1)$$

with $\rho_2 = \frac{r_k}{1-\lambda}$. This is the bivariate Chi-square distribution in

the equicorrelated case with known correlations. Hence,

$$\left(\frac{\text{MSB (adj)}}{\text{MST (adj)}}, \frac{\text{MSE}}{\text{MST (adj)}} \right) \sim \text{Biv } F (b-1, t-1, t-1, bk-b-t+1)$$

with $\rho_2 = \frac{r_k}{1-\lambda}$ can be used to find the critical points for the SANOVA

test. The above notation for the bivariate F distribution is analogous to

that of the bivariate Chi-square distribution.

In this example the canonical correlations are equal. Had they

not been so, ρ_2 would have been replaced by $\frac{1}{t-1} \sum_{j=1}^{t-1} \rho_j$ and the

same procedure would have been used to make the test.

4. A Simultaneous Test

In order to illustrate the SANOVA testing procedure an example

is taken from Cochran and Cox (1957). The authors report on an

experiment performed by the North Carolina Agricultural Experiment Station to compare 10 hybrids and 3 standard varieties. The design chosen is a SIBD with $t=13$ (10 hybrids + 3 standards) and $r=k=4$. The two ANOVA tables which can be constructed are

ANOVA (Blocks Adjusted)

Source	d.f.	S.S.	M.S.
Treatments	12	542.67	
Blocks (adj)	12	475.27	39.61
Error	27	538.21	19.93

ANOVA (Treatments Adjusted)

Source	d.f.	S.S.	M.S.
Blocks	12	689.38	
Treatments (adj)	12	328.55	27.38
Error	27	538.22	19.93

Now to test $H_0 = H_{01} \cap H_{02}$, where

H_{01} : no block effect

and

H_{02} : no treatment effect,

one can use the bivariate F distribution. Here,

$$\left(\frac{MSE}{MST(adj)}, \frac{MSE}{MSE} \right) \sim \text{Biv } F(12, 12, 12, 27)$$

with parameter $p^2 = \frac{r-\lambda}{r-1} = \frac{16}{4-1} = \frac{16}{3}$. Thus, $p \approx 35$. Since the

difference in the values of tables A-20 and A-21 for $m=12$, $n=24$ and $m=12$, $n=30$ is only one unit in the second decimal place, table A-21 is used to obtain critical points for this test with a 10% significance level. Interpolating in this table using the reciprocals of the degrees of freedom, this test is

$$\begin{aligned} &\text{reject } H_{01} \quad \text{if } \text{MSB}(\text{adj})/\text{MSE} > 2.06 \\ &\text{reject } H_{02} \quad \text{if } \text{MST}(\text{adj})/\text{MSE} > 2.06 \end{aligned}$$

and do not reject H_0 if neither H_{01} nor H_{02} are rejected. Here

$$\text{MSB}(\text{adj})/\text{MSE} = 1.99 \quad \text{and} \quad \text{MST}(\text{adj})/\text{MSE} = 1.37$$

so one cannot reject the hypothesis of no block and no treatment effects.

CHAPTER V

SIMULTANEOUS VARIANCE COMPONENT ESTIMATION

1. Introduction

In this chapter the theory of the bivariate Chi-square distribution is applied to simultaneous variance component estimation. The general subject of variance component estimation is a topic which is receiving increased attention in the literature, but one which, due to its complexity, has few practical results available to the experimenter. The value of the bivariate Chi-square distribution becomes apparent with the applications presented in this chapter.

Recently Jensen and Jones (1969) discussed simultaneous confidence intervals and referenced several other authors who wrote on the same subject. All these articles, however, suffer from a lack of generality. Procedures utilizing the bivariate Chi-square distribution as derived in preceding chapters can be used to solve many estimation problems which previously could not be handled. In particular, the restriction of equal degrees of freedom in the marginal Chi-square random variables is no longer a handicap and one need not confine himself to the equicorrelated case.

2. Classical Estimation Technique

As an introduction to simultaneous estimation procedures con-

sider two quadratic forms q_1 and q_2 where

$$q_1 = \frac{\sigma^2 + k_1 \sigma^2}{2} \chi^2(m), \quad q_2 = \frac{\sigma^2 + k_2 \sigma^2}{2} \chi^2(m),$$

and

$$\left(\frac{q_1}{\sigma^2 + k_1 \sigma^2}, \frac{q_2}{\sigma^2 + k_2 \sigma^2} \right) \sim \text{Biv } \chi^2(m, m, m). \quad (5.1)$$

The squared canonical correlations between these two Chi-square ran-

dom variables are

$$\rho_j = \frac{\sigma^2 + k_1 \sigma^2}{\sigma^2 + k_2 \sigma^2} \quad j=1, 2$$

Here the λ_j are solutions to the determinantal equation

$$\left| AB - \lambda^2 I \right| = 0$$

where A and B are the matrices of the quadratic forms q_1 and q_2 (cf.

Chapter II). The squared canonical correlations are not known since

they are functions of the unknown variance components, but the relation-

ship $\rho_j \leq \lambda_j$ holds. Now let $\lambda_j = \lambda$ for $j=1, 2, \dots, m$. One

might consider, for example, a SBIBD with

$$q_1 = \text{SST}(\text{adj}), \quad q_2 = \text{SSB}(\text{adj})$$

$$k_1 = k_2 = \frac{r}{\lambda t}$$

$$m = t - 1$$

and

$$\gamma_2 = \frac{r}{r - \lambda}$$

Consider now $\rho_2 = \gamma_2$. Numerically integrating via the computer

one can determine a constant $c_{1-\alpha}$ so that

$$\int_{c_{1-\alpha}}^0 \int_{c_{1-\alpha}}^0 f_1 \left(\frac{\sigma_2^2 + k_1 \sigma_\alpha^2}{\rho_2^2}, \frac{\sigma_2^2 + k_2 \sigma_\beta^2}{\rho_2^2} \right) d\rho_2 = 1 - \alpha, \quad (5.2)$$

$$* \quad d \left(\frac{\sigma_2^2 + k_1 \sigma_\alpha^2}{\rho_2^2} \right) d \left(\frac{\sigma_2^2 + k_2 \sigma_\beta^2}{\rho_2^2} \right) = 1 - \alpha, \quad (5.2)$$

where $f_1 \left(\frac{\sigma_2^2 + k_1 \sigma_\alpha^2}{\rho_2^2}, \frac{\sigma_2^2 + k_2 \sigma_\beta^2}{\rho_2^2} \right)$ is the density function

of the bivariate Chi-square random variable (5.1) with parameter

$\rho_2 = \gamma_2$. It is shown in Appendix 2 that for a fixed constant $c_{1-\alpha}$,

the left hand side (l. h. s.) of (5.2) is a monotonic increasing function

of ρ_2 . In computing $c_{1-\alpha}$ in (5.2) $\rho_2 = \gamma_2$ is used. Since actually

$\rho_2 \leq \gamma_2$, the value of $c_{1-\alpha}$ determined in (5.2) is smaller than the pre-

cise value which would be obtained if ρ_2 was known exactly. But

from (5.2) with $\rho_2 = \gamma_2$,

$$\Pr \left\{ 0 < \frac{\sigma_2^2 + k_1 \sigma_\alpha^2}{\rho_1^2} < c_{1-\alpha} \right\}$$

$$0 < \frac{\sigma_2^2 + k_2 \sigma_\beta^2}{\rho_2^2} < c_{1-\alpha} \Big\} = 1 - \alpha$$

or,

$$\Pr \left\{ \begin{aligned} \frac{\sigma_2^2 + k_1 \sigma_\alpha^2}{q_1} &> \frac{c_{1-\alpha}}{q_1} \\ \frac{\sigma_2^2 + k_2 \sigma_\beta^2}{q_2} &> \frac{c_{1-\alpha}}{q_2} \end{aligned} \right\} = 1 - \alpha.$$

So if $\theta^2 \approx \chi^2$ approximate 100(1- α)% lower simultaneous con-

fidence interval on $\sigma_2^2 + k_1 \sigma_\alpha^2$ and $\sigma_2^2 + k_2 \sigma_\beta^2$ is

$$\left(\sigma_2^2 + k_1 \sigma_\alpha^2 > \frac{c_{1-\alpha}}{q_1}, \quad \sigma_2^2 + k_2 \sigma_\beta^2 > \frac{c_{1-\alpha}}{q_2} \right). \quad (5.3)$$

Define c_α by

$$\int_{-\infty}^{c_\alpha} \int_{-\infty}^{c_\alpha} f_1 \left(\frac{\sigma_2^2 + k_1 \sigma_\alpha^2}{q_1}, \frac{\sigma_2^2 + k_2 \sigma_\beta^2}{q_2} \right) d \left(\frac{\sigma_2^2 + k_2 \sigma_\beta^2}{q_2} \right) = 1 - \alpha. \quad (5.4)$$

$$* \quad d \left(\frac{\sigma_2^2 + k_1 \sigma_\alpha^2}{q_1} \right) d \left(\frac{\sigma_2^2 + k_2 \sigma_\beta^2}{q_2} \right) = 1 - \alpha. \quad (5.4)$$

It is also shown in Appendix 2 that the l. h. s. of (5.4) is a monotone

increasing function of θ^2 , so that approximate 100 (1- α)% simulta-

neous upper confidence interval on $\sigma_2^2 + k_1 \sigma_\alpha^2$ and $\sigma_2^2 + k_2 \sigma_\beta^2$ is

$$\left(0 < \sigma_2^2 + k_1 \sigma_\alpha^2 < \frac{c_\alpha}{q_1}, \quad 0 < \sigma_2^2 + k_2 \sigma_\beta^2 < \frac{c_\alpha}{q_2} \right). \quad (5.5)$$

If an independent quadratic form q_3 exists, where

$$\frac{\sigma_2^2}{q_3} \sim \chi^2(s)$$

independently of q_1 and q_2 , then defining

$$MSq_1 = \frac{q_1}{m}, \quad MSq_2 = \frac{q_2}{m}, \quad \text{and} \quad MSq_3 = \frac{q_3}{s},$$

one finds that

$$\left(\frac{MSq_1}{\sigma_2^2 + k_1 \sigma_\alpha^2}, \frac{MSq_2}{\sigma_2^2}, \frac{MSq_3}{\sigma_2^2 + k_2 \sigma_\beta^2} \right)$$

$$\sim \text{Biv } F(m, m, m, s) \quad (5.6)$$

with parameter $\rho^2 \leq \gamma^2$ as above. Then approximate $100(1-\alpha)\%$ simultaneous upper and lower confidence intervals on $\frac{\sigma_\alpha^2}{\sigma_2^2}$ and $\frac{\sigma_\beta^2}{\sigma_2^2}$ are, respectively,

$$\left(0 < \frac{\sigma_\alpha^2}{\sigma_2^2} < \frac{1}{k_1} \left[\frac{MSq_1}{f_\alpha MSq_3} - 1 \right] \right),$$

$$0 < \frac{\sigma_\beta^2}{\sigma_2^2} < \frac{1}{k_2} \left[\frac{MSq_2}{f_\alpha MSq_3} - 1 \right]$$

$$(5.7)$$

and

$$\left(\frac{1}{k_1} \left[\frac{MSq_1}{f_{1-\alpha} MSq_3} - 1 \right] < \frac{\sigma_\alpha^2}{\sigma_2^2}, \frac{1}{k_2} \left[\frac{MSq_2}{f_{1-\alpha} MSq_3} - 1 \right] < \frac{\sigma_\beta^2}{\sigma_2^2} \right)$$

$$(5.8)$$

where f_α and $f_{1-\alpha}$ are bivariate F critical points determined with $\rho^2 = \gamma^2$ and defined similarly to c_α and $c_{1-\alpha}$ of the bivariate Chi-square distribution.

The above procedure can be used in any problem in which the Chi-square random variables have equal degrees of freedom and all squared canonical correlations equal and non-zero. When the degrees of freedom are unequal variations of the density function, which were derived in Chapter III, can be used with upper and lower critical points suitably defined. The confidence intervals are approximate, but from the discussion in Chapter VI the intervals should be close to the true intervals unless ρ^2 is much less than γ^2 . In situations where the squared correlations are unequal and perhaps some are zero, conservative intervals can be found using the appropriate form of the equicorrelated bivariate Chi-square density with the approximation

$$\rho^2 = \frac{1}{m} \sum_{j=1}^m \gamma_j^2 .$$

3. Estimating Individual Variance Components in the Equi-

Correlated Case

Continuing the discussion of the previous section, a method for finding confidence intervals on the individual variance components σ_a^2 and σ_b^2 will now be presented. The technique is due to Williams (1962), but it does not seem to have made much of an impact in the literature. It is, nevertheless, a valuable tool.

To illustrate Williams' procedure the Chi-square random

variables will again be of the form (5.1). Variations of these results

can again be made for unequal degrees of freedom and approximations

used when the correlations are not all identical. Begin with (5.4) and c_α computed for $\rho = \gamma_2$. Then

$$\Pr \left\{ \frac{q_1}{2} \frac{\sigma_2^2 + k_1 \sigma_\alpha^2}{2} > c_\alpha, \frac{q_2}{2} \frac{\sigma_2^2 + k_2 \sigma_\beta^2}{2} > c_\alpha \right\} \approx 1 - \alpha$$

or

$$\Pr \left\{ 0 < \sigma_2^2 < \frac{1}{k_1} \left[\frac{c_\alpha}{q_1} - \sigma_2^2 \right], 0 < \sigma_2^2 < \frac{1}{k_2} \left[\frac{c_\alpha}{q_2} - \sigma_2^2 \right] \right\}$$

$$0 < \sigma_2^2 < \frac{1}{k_2} \left[\frac{c_\alpha}{q_2} - \sigma_2^2 \right] \mid \sigma_2^2 \mid \left\{ \approx 1 - \alpha. \right. \quad (5.9)$$

Denoting the interval $(0, \frac{1}{k_1} \left[\frac{c_\alpha}{q_1} - \sigma_2^2 \right])$ by $I_{1, q_1}(\sigma_2^2)$ and

$(0, \frac{1}{k_2} \left[\frac{c_\alpha}{q_2} - \sigma_2^2 \right])$ by $I_{2, q_2}(\sigma_2^2)$, then (5.9) can be written as

$$\Pr \left\{ \sigma_2^2 \in I_{1, q_1}(\sigma_2^2), \sigma_2^2 \in I_{2, q_2}(\sigma_2^2) \mid \sigma_2^2 \right\} \approx 1 - \alpha. \quad (5.10)$$

A second probability statement using the bivariate F distribution (5.6)

is

$$\Pr \left\{ \frac{MSq_1}{2} \frac{\sigma_2^2 + k_1 \sigma_\alpha^2}{2} > f_\alpha, \frac{MSq_2}{2} \frac{\sigma_2^2 + k_2 \sigma_\beta^2}{2} > f_\alpha \right\} \approx 1 - \alpha.$$

$$\Pr \left\{ \sigma_2^\alpha e_{I_2, q_1}(\sigma_2), \sigma_2^\beta e_{I_2, q_2}(\sigma_2) \mid \sigma_2 \right\} \approx 1 - \alpha \quad (5.11)$$

where

$$I_{2, q_1}(\sigma_2) = (0, \frac{1}{k_1} \left[\frac{MS_{q_1}^\alpha}{MS_{q_3}^\alpha} - 1 \right] \sigma_2) \quad \text{and} \quad I_{2, q_2}(\sigma_2) = (0, \frac{1}{k_2} \left[\frac{MS_{q_2}^\alpha}{MS_{q_3}^\alpha} - 1 \right] \sigma_2)$$

(5.10) and (5.11) are joint confidence statements on the individual variance components σ_2^α and σ_2^β given that σ_2 is known. Now,

$$\begin{aligned} & \Pr \left\{ \sigma_2^\alpha e_{I_1, q_1}(\sigma_2), \sigma_2^\beta e_{I_1, q_2}(\sigma_2) \mid \sigma_2 \right\} \\ &= \Pr \left\{ \sigma_2^\alpha e_{I_2, q_1}(\sigma_2), \sigma_2^\beta e_{I_2, q_2}(\sigma_2) \mid \sigma_2 \right\} + \Pr \left\{ \sigma_2^\alpha e_{I_2, q_1}(\sigma_2), \sigma_2^\beta e_{I_2, q_2}(\sigma_2) \mid \sigma_2 \right\} \\ &\quad - \Pr \left\{ \sigma_2^\alpha e_{I_1, q_1}(\sigma_2), \sigma_2^\beta e_{I_1, q_2}(\sigma_2) \mid \sigma_2 \right\} \cup \left[\sigma_2^\alpha e_{I_2, q_1}(\sigma_2), \sigma_2^\beta e_{I_2, q_2}(\sigma_2) \mid \sigma_2 \right] \\ &\approx (1 - \alpha) + (1 - \alpha) - (1 - \Pr \left\{ \sigma_2^\alpha e_{I_1, q_1}(\sigma_2) \right\}) \end{aligned}$$

This results in (5.11):

$$U \left[\sigma_2 \notin I_{1,q_1}(\sigma_2) \right] \cup \left[\sigma_2 \notin I_{2,q_2}(\sigma_2) \right] \geq 1 - 2\alpha.$$

It is important to note at this point that the lower bound on this conditional probability, $1 - 2\alpha$, is independent of the parameter σ_2 . In

addition, the interval $I(\sigma_2)$ formed by the intersection of the boundaries of $I_{1,q_1}(\sigma_2)$ and $I_{2,q_2}(\sigma_2)$ and the origin is independent of σ_2

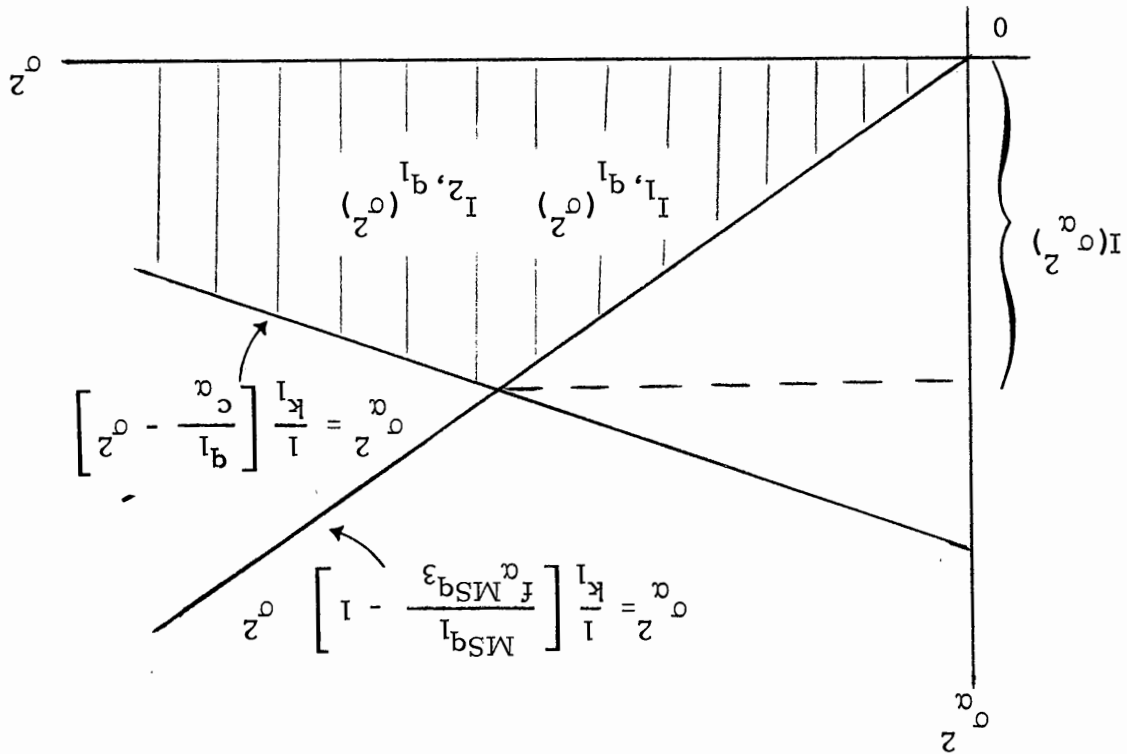
and will always include σ_2 whenever $I_{1,q_1}(\sigma_2)$ and $I_{2,q_2}(\sigma_2)$ include

σ_2 . The same statement can be made about the interval $I(\sigma_2)$

formed by the intersection of the boundaries of $I_{1,q_1}(\sigma_2)$ and $I_{2,q_2}(\sigma_2)$

and the origin

To make these ideas clearer, consider the following diagram:



The boundaries of $I_1, q_1(\sigma^2)$ and $I_2, q_1(\sigma^2)$ intersect when

$$\frac{1}{k_1} \left[\frac{q_1}{c} - \sigma^2 \right] = \frac{1}{k_1} \left[\frac{f_{MSq_1}^\alpha}{MSq_1} - 1 \right] \sigma^2,$$

i. e. when

$$\sigma^2 = MSq_3 f_{m/c}^\alpha.$$

Inserting this value of σ^2 into the formula for $I_1, q_1(\sigma^2)$ the value of

σ^2 at the intersection of the boundaries, and therefore, its maximum value is

$$\sigma^2 = \frac{m}{k_1 c} \left[MSq_1 - f_{MSq_3}^\alpha \right].$$

Thus

$$I(\sigma^2) = \left(0, \frac{m}{k_1 c} \left[MSq_1 - f_{MSq_3}^\alpha \right] \right).$$

Similarly one finds

$$I(\sigma^2) = \left(0, \frac{m}{k_2 c} \left[MSq_2 - f_{MSq_3}^\alpha \right] \right).$$

Note here that, as mentioned above, neither $I(\sigma^2)$ nor $I(\sigma^2)$ depend

on σ^2 . One can, therefore, write

$$\Pr \left\{ \sigma^2 \in I(\sigma^2), \sigma^2 \in I(\sigma^2) \right\} \geq 1-2\alpha$$

and this probability statement is independent of the nuisance parameter

σ^2 . Hence a joint confidence interval on σ^2 and σ^2 with confidence

approximately 100(1-2 α)% is

$$(\sigma^2 \in I(\sigma^2), \sigma^2 \in I(\sigma^2)).$$

4. Confidence Limits by Partitioning the Error Sum of Squares

The techniques used in the previous section cannot be applied to more general situations than those described therein. For example, consider a general linear regression model:

$$\bar{Y} = u\bar{1} + X_B\bar{\beta} + X_T\bar{\epsilon} + e$$

where

$\bar{Y}$ is an $n \times 1$ vector of observed values

u is an unknown constant

$\bar{1}$ is an $n \times 1$ vector of ones

X_B is an $n \times b$ matrix of zeros and ones such that each

row vector contains a single one and $b-1$ zeros

X_T is an $n \times t$ matrix of zeros and ones such that each

row vector contains a single one and $t-1$ zeros

$\bar{\beta}$ is a $b \times 1$ vector of independent random variables

with $\beta_j \sim N(0, \sigma_{\beta_j}^2)$, $j = 1, 2, \dots, b$

$\bar{T}$ is a $t \times 1$ vector of independent random variables

with $T_j \sim N(0, \sigma_{T_j}^2)$, $j = 1, 2, \dots, t$

$\bar{\epsilon}$ is an $n \times 1$ vector of independent random variables

with $\epsilon_j \sim N(0, \sigma_{\epsilon_j}^2)$, $j = 1, 2, \dots, n$.

Again it is desired to estimate $\sigma_{\epsilon_j}^2$ and $\sigma_{\beta_j}^2$ simultaneously. Making

the transformations

$$Z_T = X_T^T - X_B^T (X_B^T X_B)^{-1} X_B^T X_T^T$$

$$\text{and } Z_B = X_B^T - X_T^T (X_T^T X_T)^{-1} X_T^T X_B^T$$

then the quadratic forms

$$\bar{Y}' Z_T' (Z_T' Z_T)^{-1} Z_T' \bar{Y}$$

and

$$\bar{Y}' Z_B' (Z_B' Z_B)^{-1} Z_B' \bar{Y}$$

are, as in Chapter IV, used to eliminate the confounding of treatment and block effects. Continuing as in Chapter IV, there exist semi-

orthogonal matrices E ($t - 1 \times t$) and F ($b - 1 \times b$) such that

$$E' Z_T' E = D_1 = \text{diag}(k_1, k_2, \dots, k_{t-1})$$

and

$$F' Z_B' F = D_2 = \text{diag}(\ell_1, \ell_2, \dots, \ell_{b-1})$$

where the k_j and ℓ_j are the latent roots of $Z_T' Z_T$ and $Z_B' Z_B$, respectively. Here all the k_j are not identical nor are all the ℓ_j since this

case was considered in the previous section. Defining

$$\bar{Z} = D_1^{-1/2} E' Z_T' \bar{Y} \quad \text{and} \quad \bar{W} = D_2^{-1/2} F' Z_B' \bar{Y},$$

then

$$\bar{Z}' \bar{Z} = \bar{Y}' Z_T' (Z_T' Z_T)^{-1} Z_T' \bar{Y} \quad \text{and}$$

$$\bar{W}' \bar{W} = \bar{Y}' Z_B' (Z_B' Z_B)^{-1} Z_B' \bar{Y}.$$

Also, since

$$\bar{Z} \sim N(0, \sigma^2 I + \sigma^2 D_1)$$

and

$$\bar{W} \sim N(0, \sigma^2 I + \sigma^2 D_2),$$

then it follows that

$$t - 1 \sum_{j=1}^n \frac{\sigma_j^2 + k_j \sigma_j^t}{z_j^2} \sim \chi^2_{t-1}$$

and

$$b - 1 \sum_{j=1}^n \frac{\sigma_j^2 + \ell_j \sigma_j^b}{w_j^2} \sim \chi^2_{b-1} \quad (5.13)$$

Nothing again that the k_j and ℓ_j are not all equal, it is clear from

(5.13) that in this general situation there is no direct method of

extracting σ_j^t and σ_j^b from the transformed quadratic forms and

determining confidence intervals as in the previous section. A

further transformation due to Thomas (1966), however, does enable

one to place confidence intervals on σ_j^t and σ_j^b .

In order to apply Thomas' transformation, suppose one has

three quadratic forms

$$q_1 = \sum_{m=1}^n \frac{z_m^2 + k_m \sigma_m^a}{z_m^2}, \quad q_2 = \sum_{m=1}^n \frac{w_m^2 + \ell_m \sigma_m^b}{w_m^2},$$

and

$$q_3 = \frac{1}{\sum_{s=1}^n} \sum_{j=1}^n \frac{\sigma_j^2}{v_j^2},$$

where

$$\begin{pmatrix} \bar{z} \\ \bar{w} \end{pmatrix} \sim N \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{bmatrix} D_1 & I_{\sigma^2} \\ I_{\sigma^2} & D_2 \end{bmatrix}$$

$$\bar{v} \sim N(\bar{0}, \sigma^2 I) \quad \text{independently of } \bar{z} \text{ and } \bar{w}$$

$$D_1 = \sigma^2 I + \sigma^2 K, \quad K = \text{diag}(k_1, k_2, \dots, k_m),$$

$$k_j > 0 \quad j = 1, 2, \dots, m$$

$$D_2 = \sigma^2 I + \sigma^2 L, \quad L = \text{diag}(\ell_1, \ell_2, \dots, \ell_m),$$

$$\ell_j > 0 \quad j = 1, 2, \dots, m$$

$$I = \gamma I, \quad \gamma \neq 0$$

$$s > m$$

and not all the k_j nor ℓ_j are identical. Variations of this model and

approximations if $I = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_m)$ are immediate and will not be discussed here.

Thomas suggests randomly selecting at most $2m-2$ of the v_j

$$f_1 = z^{(1)}$$

$$f_j = \frac{1}{\sqrt{z^2 + 1}} (z_j + c_j v_j) \quad j = 2, 3, \dots, m$$

where $z^{(1)}$ is the variable in $\bar{z}$ corresponding to the smallest k_j . If

there are n variables in $\bar{z}$ that have the smallest k_j ($n < m$) then one will only need to select $m-n$ of the v_j to construct f_j as above. The

constants c_j are chosen so that

$$\text{Var}(f_j) = \sigma^2 + \delta \sigma^2 \quad j = n+1, n+2, \dots, m,$$

and δ should be chosen as large as possible so that the confidence intervals to be formed will be as small as possible. Now,

$$\text{Var}(\xi_j) = \frac{c_j^2 + 1}{1} \left[\text{Var}(z_j) + c_j^2 \text{Var}(v_j) \right]$$

$$= \frac{c_j^2 + 1}{1} \left[\sigma_z^2 + k_j^2 \sigma_\alpha^2 + c_j^2 \sigma_z^2 \right]$$

$$= \sigma_z^2 + \frac{c_j^2 + 1}{k_j^2} \sigma_\alpha^2.$$

Then one must have

$$\delta = \frac{k_j^2}{c_j^2 + 1}$$

or,

$$c_j^2 = \left(\frac{k_j^2}{\delta} - 1 \right)^{-1/2}$$

so that a constraint on δ is

$$\delta > k_j^2 \quad j = n+1, n+2, \dots, m.$$

Hence one wishes to choose a value for δ that is as large as possible subject to this constraint. Note that if one chooses $\delta = k^{(1)}$, where $k^{(1)}$ is the smallest k_j , then defining $c_j^2 = 0$ for $j = 1, 2, \dots, n$ one

can write

$$\xi_j = \frac{\sqrt{c_j^2 + 1}}{1} (z_j + c_j v_j) \quad j = 1, 2, \dots, m$$

where

$$c_j = 0 \quad j = 1, 2, \dots, n$$

$$c_j = + \left(\frac{k_j}{k_j^{(1)}} - 1 \right)^{-1/2} \quad j = n+1, n+2, \dots, m$$

and

$$\text{Var}(\xi_j) = \sigma^2 + k_j^{(1)} \sigma_a^2 \quad j = 1, 2, \dots, m.$$

Analogously, if $p =$ multiplicity of $\lambda_j^{(1)}$ use $m-p$ of the randomly

selected v_j and form

$$\eta_j = \frac{1}{\sqrt{d_j^2 + 1}} (w_j + d_j v_j) \quad j = 1, 2, \dots, m$$

where

$$d_j = 0 \quad j = 1, 2, \dots, p$$

$$d_j = + \left(\frac{\lambda_j^{(1)}}{\lambda_j} - 1 \right)^{-1/2} \quad j = p+1, p+2, \dots, m,$$

then

$$\text{Var}(\eta_j) = \sigma^2 + \lambda_j^{(1)} \sigma_g^2.$$

The best method for selecting the v_j for the ξ_j and η_j is open for

discussion and will not be considered here. The above method is

but one of the possible ways to do so.

Now one finally has

$$\begin{pmatrix} \bar{\eta} \\ \bar{\xi} \end{pmatrix} \sim N \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} (\sigma_2^2 + k^{(1)} \sigma_\alpha^2) I & R \\ R & (\sigma_2^2 + \ell^{(1)} \sigma_\beta^2) I \end{bmatrix}$$

where $R = \text{diag}(r_1, r_2, \dots, r_m)$. Note that if R is not diagonal an

orthogonal transformation can be used to make it so. Hence one

has three quadratic forms

$$q_1^* = \frac{\sum_{j=1}^m \xi_j^2}{2} + \frac{\sum_{j=1}^m \eta_j^2}{2}, \quad q_2 = \frac{\sum_{j=1}^m \sigma_2^2}{2} + \frac{\sum_{j=1}^m \sigma_\beta^2}{2} \quad \text{and}$$

$$q_3 = \frac{\sum_{j=1}^m v_j^2}{2}$$

where q_3 contains the $s^* v_j$ not used to form either q_1 or q_2 . One

can now use all the previous results to obtain desired confidence

intervals.

5. Confidence Intervals Without Partitioning the Error Sum

of Squares

There are two major drawbacks of the procedure presented in

the previous section for estimating individual variance components.

The first drawback is that s^* may turn out to be small and thus q_3

may be a poor estimate of σ_2^2 . Perhaps a more important difficulty

is that $k^{(1)}$ or $\ell^{(1)}$ may turn out to be very small and thus the confidence

intervals, which are a function of $k^{(1)}$ and $\ell^{(1)}$, would be very large.

Returning to the quadratic forms

$$q_1 = \sum_{j=1}^m \frac{z_j^2}{z_j^2 + k_j^2 \sigma^2} \quad \text{and} \quad q_2 = \sum_{j=1}^m \frac{z_j^2}{z_j^2 + \ell_j^2 \sigma^2 + \ell_j^2 \sigma^2} ,$$

where not all the k_j and ℓ_j are identical, if m is sufficiently large a transformation due to Mason and Webster (1972) can be used to remove the above hindrances. Consider first q_1 and order the m

latent roots

$$k^{(1)} \leq k^{(2)} \leq k^{(3)} \leq \dots \leq k^{(m)} .$$

There will be at least one strict inequality here, so define k_0 such that at most $\frac{m}{2}$ of the k_j are greater than k_0 and at most $\frac{m}{2}$

are less than k_0 . Consider first the case where k_0 has multiplicity

of at most one in the ordered k_j . One can then write

$$\text{Var}(\bar{z}) = I \sigma^2 + K \sigma^2$$

$$= (I \sigma^2 + K \sigma^2) I + M \sigma^2$$

where

$$K = \text{diag}(k^{(1)}, k^{(2)}, \dots, k^{(m)})$$

$$M = \text{diag}(k^{(1)} - k_0, k^{(2)} - k_0, \dots, k^{(m)} - k_0)$$

$$\text{or, } M = \text{diag}(k_1^*, k_2^*, \dots, k_m^*)$$

with $k_j^* = k_{(j)} - k_0$ $j = 1, 2, \dots, m$. Pair the positive and

negative k_j^* :

$$\begin{aligned} & (k_{n1}^*, k_{p1}^*) \\ & \vdots \\ & (k_{n2}^*, k_{p2}^*) \\ & \vdots \\ & (k_{nr}^*, k_{pr}^*) \end{aligned}$$

where $r \leq \frac{m}{2}$ if m is even or $r \leq \frac{m-1}{2}$ if m is odd, equality

occurring in many instances (for example, when all the $k_{(j)}^*$ are distinct). Now from the negative k_{nj}^* and the positive k_{pj}^* form

$$R_{nj, pj} = \left(\frac{|k_{nj}^*|}{k_{pj}^*} \right)^{1/2} \quad j = 1, 2, \dots, r$$

where the k_{nj}^* and k_{pj}^* are paired above so that all the $R_{nj, pj}$ are

as close to unity as possible. Mason and Webster then form the

vectors

$$\underline{c}_j = (1 + R_{nj, pj}^2)^{-1/2} \left[\alpha_{pj}^* + R_{nj, pj} \alpha_{nj}^* \right] \quad j = 1, 2, \dots, r$$

where α_{pj}^* and α_{nj}^* are the latent vectors of M corresponding to

k_{pj}^* and k_{nj}^* . Note that

$$\underline{c}_i^* \underline{c}_j^* = 0 \quad \text{for all } i \neq j \quad \text{and} \quad \underline{c}_j^* \underline{c}_j^* = 1, \quad j = 1, 2, \dots, r.$$

Also, $C'_1 MC = \Phi$, where $C = (\bar{c}_1, \bar{c}_2, \dots, \bar{c}_r)$. Finally, make

the transformation

$$\bar{u}_z = C'_1 \bar{z} \sim N(0, (\sigma^2 + k_0 \sigma^2_2) I)$$

so that

$$q_1 = \frac{\bar{u}_z' \bar{u}_z}{\sigma^2 + k_0 \sigma^2_2} \sim \chi^2_2(r)$$

By a similar series of transformations one finds

$$q_2 = \frac{\bar{u}_w' \bar{u}_w}{\sigma^2 + k_0 \sigma^2_2} \sim \chi^2_2(r^*)$$

Once again one has three quadratic forms q_1 , q_2 , and q_3 and the

procedures discussed in sections 2 and 3 can be used to determine

appropriate confidence intervals.

If the median value of the $k^{(j)}$ has multiplicity greater than one

a slight modification to the above procedure must be used. Suppose

$$k^{(1)} \leq k^{(2)} \leq \dots \leq k^{(a)} < k_0 = k_0 = \dots = k_0$$

$$< k^{(a+b+1)} \leq \dots \leq k^{(m)},$$

i. e. the median value has multiplicity b . Then for the z_j corresponding

to the median value k_0 ,

$$\text{Var}(z_j) = \sigma^2 + k_0 \sigma^2_2 \quad j = a+1, a+2, \dots, a+b$$

If one uses Mason and Webster's procedure there will be an additional

c independent variables in $\bar{u}_z$ (where $c = \min \{a, m-a-b\}$) so that

one can form

$$p_0^1 = \left(\sum_{j=1}^{a+b} z_j^j + \sum_{i=1}^n z_i^i - z \right) / \left(\sigma^2 + k_0 \sigma^2 \right) \sim \chi^2_{(b+c)}.$$

The same procedure can be applied to the variables in $\overline{w}$ if ℓ_0

has multiplicity greater than one.

The procedures used in this and the previous sections are approx-

imate, but from the discussion in Chapter VI they should be good unless ν^2 is large and ρ^2 is considerably smaller than ν^2 .

CHAPTER VI

CRITICAL POINTS OF THE BIVARIATE CHI-SQUARE AND BIVARIATE F DISTRIBUTIONS

1. Introduction

This chapter contains computing techniques for finding critical points of the bivariate Chi-square and bivariate F distributions. One should bear in mind that the main objective of this research is not to construct tables but to develop theory and applications of the bivariate Chi-square and bivariate F distributions. Thus, only

selective tables have been presented in this paper. The discussion contained in sections 3, 4, and 5 of this chapter, moreover, reveals that complete tables are not needed for many applications. In most practical cases the exact critical points can be closely estimated using values from the univariate tables - a result of great value.

2. Approximating the Bivariate Chi-square Density

The first difficulty encountered when calculating bivariate

Chi-square or bivariate F critical points is the approximating of the density function. The density function of the bivariate Chi-square

distribution with equal degrees of freedom and all squared correlations

non-zero and equal is

$$f_1(q_1, q_2) = \frac{\sum_{j=0}^{\infty} \Gamma\left(\frac{m}{2}\right) 2^{m/2} (1-\rho)^j}{1} \left[\frac{\rho}{2(1-\rho)^2} \right]^{2j} (q_1 q_2)^{\frac{m}{2} + j - 1} e^{-\frac{q_1 + q_2}{2(1-\rho)^2}}. \quad (6.1)$$

This is equation (3.1) of Chapter III. The density function for unequal

degrees of freedom with all squared correlations non-zero and equal is given by (3.2) and denoted by $f_2(q_1, q_2)$. The density function for

unequal degrees of freedom with some squared correlations zero and the remaining ones equal is given by (3.3) and denoted by $f_3(q_1, q_2)$.

In all these cases if the correlations are not identical an approximation to the true density function can be made using (3.1), (3.2), or

(3.3), with ρ^2 replaced by

$$\frac{1}{k} \sum_{j=1}^k \rho_j^2.$$

In each of the above density functions there is at least one infinite

summation. The density function is only valuable if the infinite

summation can be replaced by a finite one and the number of terms

needed for a good approximation is not excessive. In order to

determine the number of terms necessary to obtain a good approxi-

mation of (6.1), the density function was integrated over all its

possible values yielding

$$\int_{-\infty}^0 \int_{-\infty}^0 f_1(q_1, q_2) p_{q_1} p_{q_2} = \frac{\Gamma\left(\frac{m}{2}\right)}{(1-p)^{m/2}} \Gamma\left(\frac{m}{2}\right)$$

$$* \sum_{j=0}^{\infty} \frac{\Gamma(j+1)}{\Gamma\left(\frac{m}{2} + j\right) p^j} \quad (6.2)$$

Since this expression reduces to unity, the number of terms used to approximate the density (6.1) for various values of m and p was taken to be the smallest number of terms needed in (6.2) to obtain a value for the right side of the equation of at least 0.999. The

number of terms needed, for some values of m and p , to give the desired accuracy is shown in Table A-1 of Appendix I. For the

densities $f_2(q_1, q_2)$ and $f_3(q_1, q_2)$ the value from Table A-1 for the largest of the degrees of freedom in each case was used in all the summations. This was done for ease in the programming and to

ensure a good approximation.

3. Computing Critical Points of the Bivariate Chi-square

Distribution

For the bivariate Chi-square distribution two methods are

used to compute critical points. If the density is of the form (3.1),

i. e. equal degrees of freedom in the marginal Chi-squares, then

the critical points for q_1 and q_2 are taken to be identical. Thus, for

fixed α , a constant c is found such that

$$\int_c^0 \int_c^0 f_1(q_1, q_2) dq_1 dq_2 = 1 - \alpha \quad (6.3)$$

The left side of (6.3) is numerically integrated and values of c are determined by an iterative procedure. The criterion used for

acceptance of a value for c is that the left side of (6.3), when numerically integrated, differs from $(1 - \alpha)$ by less than 10^{-4} . The program was checked for $p = 0$ and $p = 1$ against the appropriate values in a

univariate Chi-square table. Tables A-2, A-3, and A-4 list critical points for various values of m and p , and for $\alpha = 0.025, 0.05$, and 0.10 . A quick examination of these tables reveals that for small

values of p , say $p = 0.0$ to $p = 0.4$, the critical points change very little. Indeed, consider the following table, partially reproduced from Table A-3 (values for $p = 0.0$ and $p = 1.0$ obtained from Ostle

(1963)). From this table one can detect that the critical points change radically only for large values of p ; the greatest differences occur-

ring between $p = 0.9$ and $p = 1.0$. Hence, especially for testing purposes, good approximate critical points for the bivariate Chi-square distribution can be obtained from the univariate Chi-square tables with α (univariate) $= 1 - \sqrt{1 - \alpha}$ (equivalent to $p = 0.0$) or α (univariate) $= \alpha$ (equivalent to $p = 1.0$).

When the degrees of freedom of q_1 and q_2 differ, in which

case the density takes the form (3.2) or (3.3), then the critical points

Table 6.1

Upper 5% Critical Points for the Bivariate
Chi-Square Distribution in the Form of Eq. (6.1)

m \ p				
	1	2	3	4
0.0	5.02	7.38	9.35	11.1
0.1	5.00	7.35	9.32	11.11
0.2	4.99	7.34	9.31	11.10
0.3	4.97	7.32	9.29	11.08
0.4	4.94	7.29	9.26	11.05
0.5	4.89	7.24	9.21	11.00
0.6	4.84	7.18	9.14	10.93
0.7	4.75	7.09	9.05	10.83
0.8	4.63	6.95	8.90	10.67
0.9	4.45	6.73	8.65	10.40
1.0	3.84	5.99	7.81	9.49

are not chosen identical. Suppose $q_1 \sim \chi^2(m)$ and $q_2 \sim \chi^2(m+n)$, so the density is given by (3.2), and let $d^\alpha(v)$ represent the upper 100 α % critical point of a univariate Chi-square random variable with v degrees of freedom. Then the two desired critical points are c_1 and c_2 where

$$c_2 = \gamma c_1, \quad \gamma = d_{\delta}(m+n) / d_{\delta}(m), \quad \delta = 1 - \sqrt{1 - \alpha}.$$

This method of choosing the critical points is chosen so that for

$$\rho = 0,$$

$$c_1 = d_{\delta}(m) \quad \text{and} \quad c_2 = d_{\delta}(m+n),$$

which should be the case for independent statistics. Table A-5 lists 5% upper critical points for the density (3.2) for selected values of

$m+n$, m , and ρ . Again, even for ρ of moderate size there is little

difference from the univariate critical points for $\alpha = .025$. Critical

points for the density (3.3) are not tabled, but the computations are

performed similarly to the above case. All values in tables in

Appendix I are accurate to three significant figures with the exception

of a few values which may be off by at most two units in the third

significant figure.

4. Critical Points of the Bivariate F Distribution

Computation of critical points of the bivariate F distribution

follows directly from the discussion of the previous section. Note

that

$$\Pr \left\{ F_1 \leq c, F_2 \leq c \right\} = \Pr \left\{ \frac{q_1/m_1}{q_3/s} \leq c, \frac{q_2/m_2}{q_3/s} \leq c \right\}$$

the critical points for small and even moderate values of p . units in the second decimal place. Again note the small changes in points are identical, but in a few cases they differ by at most two Krishnaiah and Armitage (1965). In nearly all cases the critical special case $m = 1$, these values can be checked against those of are presented in tables A-6 through A-23 of Appendix I. For the

$$(F_1, F_2) \sim \text{BivF}(m, m, m, s) \quad (6.4)$$

Critical points for the bivariate F distribution where

$$\text{Expectation, } E[MSq_i] / E[MSq_j] = 1, i = 1, 2.$$

numerator and denominator of any F statistic have the same ex- critical points of F_1 and F_2 are taken to be identical since the variate Chi-square density over proper limits. In all cases the lated by integrating the appropriate Chi-square density and a uni- Thus critical points for the bivariate F distribution can be calcu-

$$* f(q_1, q_2) h(q_3) dq_1 dq_2 dq_3 .$$

$$= \int_{-\infty}^0 \int_{m_2 c q_3 / s}^0 \int_{m_1 c q_3 / s}^0$$

$$\left\{ q_2 \leq \frac{m_2 c q_3}{s} \right\}$$

$$= \Pr \left\{ q_1 \leq \frac{m_1 c q_3}{s} \right\} ,$$

5. Estimating Bivariate Chi-square Critical Points from

Univariate Tables

In this section a method for obtaining approximate bivariate Chi-square critical points will be suggested. The method was discovered empirically, but gives good results over a wide range of the parameter p . The same procedure can be used for estimating bivariate F critical points with the same restrictions on p as described below.

In many testing problems, knowledge of exact critical points are not necessary since the test statistic is often much larger or much smaller than the critical point. In this case two bounds c_λ and c_u on the true critical point c will suffice. Then if the test statistic is larger than c_u the hypothesis is rejected whereas if it is smaller than c_λ the hypothesis is not rejected. In some testing problems and in all estimation problems an exact or nearly exact critical point is demanded. So for the problems and examples considered in previous chapters comprehensive bivariate Chi-square tables are needed. Obviously it would be a very arduous and expensive task to compile all the tables that would be necessary for any estimation problem. In lieu of this, a simple technique will enable one to often obtain a very good estimate of the true critical point for a wide range of the parameter p .

Tables of the univariate Chi-square distribution are readily

available and it is desired to estimate critical points of the bivariate Chi-square distribution with parameter ρ^2 from the univariate

tables. It is true for a fixed α and the following probability state-

ments

$$(6.5) \quad \left\{ \begin{array}{l} \Pr \left\{ q_1 < c_\lambda, q_2 < c_\lambda \mid \rho^2 = 1 \right\} = 1 - \alpha \\ \Pr \left\{ q_1 < c, q_2 < c \mid \rho^2 \neq 0, 1 \right\} = 1 - \alpha \\ \Pr \left\{ q_1 < c_u, q_2 < c_u \mid \rho^2 = 0 \right\} = 1 - \alpha \end{array} \right.$$

that $c_\lambda \leq c \leq c_u$ for $0 \leq \rho^2 \leq 1$. This result is easily verified by

the proof in Appendix 2 that $\Pr \left\{ q_1 \leq c, q_2 \leq c \right\}$ is a mono-

tone increasing function of ρ for a fixed c . Thus from (6.5) if, for example, $\alpha = .05$, c_λ is an upper 5% critical point of a univariate

Chi-square random variable while c_u is an upper 2.5% critical point.

Consider now the critical points of Table A-3. If one plots

the critical points for any chosen m versus the respective values of

ρ^2 one finds that the curve is quadratic. Then a simple relation be-

tween the critical points for various values of ρ is

$$(6.6) \quad c(\rho) = c_\lambda + a(\rho)^2$$

where c_λ is the univariate upper 5% critical point. Note that in

equation (6.6) $c(0) = c_\lambda$ as desired. The other constraint on (6.6)

is that $c(1) = c_u$.

This implies that

$$c(1) = c_u = c_\lambda + a$$

or,

$$a = c_u - c_\lambda$$

where c_u is the upper 2-l/2% critical point. Thus an estimate of the

bivariate Chi-square critical points for a fixed α and arbitrary ρ is

$$c(\rho) = c_\lambda + (c_u - c_\lambda) (\rho)^2 \quad (6.7)$$

where c_λ is the upper 100 α % critical point with the appropriate

degrees of freedom, and c_u is the upper 100 (1 - $\sqrt{1-\alpha}$) % critical

point with the same degrees of freedom. Note in the general case,

i. e. where the squared canonical correlations are not all identical,

$$\text{one would replace } \rho^2 \text{ in (6.7) with } \rho \cdot \frac{\sum_{j=1}^k \rho_j^2}{2} = \frac{k}{1} \cdot \rho \cdot \frac{\sum_{j=1}^k \rho_j^2}{2} .$$

In order to check equation (6.7) the calculated critical points

for the bivariate Chi-square distribution of equation (3.1) for

$m = 1, 8, \text{ and } 20$ are compared with the estimated values. The cal-

culated values are taken from Table A-3. These values are pre-

sented in Table 6.2. From this table note the remarkable accuracy

of equation (6.7) in estimating critical points for values of ρ as large

as 0.7. For small values of m the estimated critical points are

close to the calculated ones even for larger values of ρ .

Table 6.2

Comparison of Calculated and Estimated Bivariate Chi-square
Critical Points

ρ	$m = 1$		$m = 8$		$m = 20$	
	Calculated	Estimated	Calculated	Estimated	Calculated	Estimated
0.1	5.00	5.02	17.50	17.50	34.12	34.20
0.2	4.99	5.02	17.48	17.50	34.10	34.20
0.3	4.97	5.01	17.47	17.48	34.08	34.18
0.4	4.94	4.99	17.43	17.45	34.04	34.13
0.5	4.89	4.95	17.38	17.37	33.98	34.02
0.6	4.84	4.87	17.30	17.24	33.88	33.84
0.7	4.75	4.74	17.18	17.02	33.73	33.53
0.8	4.63	4.54	16.99	16.68	33.47	33.05
0.9	4.45	4.25	16.66	16.19	33.03	32.36

This procedure can also be used to estimate the critical points of the bivariate Chi-square distribution of equations (3.2) and (3.3) with only slight modifications. Since in both these cases the critical points are of the form c and γc , where γ is a constant defined in section 3, simply determine c by the above method and multiply it by γ to obtain the other critical point. Bivariate F critical points can also be found using above procedure.

CHAPTER VII

CONCLUSION

This study has attempted to develop the theory of the bivariate Chi-square distribution so that it can be a useful instrument for the experimenter in analyzing data. In the process of developing a practical theory an approximation to the general case (unequal squared canonical correlations) using the equicorrelated case by replacing ρ^2

with

$$\frac{1}{k} \sum_{j=1}^k \rho_j^2$$

is suggested. Moment considerations reveal that this approximation should be excellent and so the need to consider the much more complex general case has been removed.

Concentrating on the equicorrelated bivariate Chi-square distribution, new forms of the density function were derived in Chapter III. These new forms are concerned with the case where the marginal Chi-square random variables have unequal degrees of freedom.

Critical points of the bivariate Chi-square and the related bivariate F distributions are discussed in Chapter VI. Some representative tables of these critical points are presented in Appendix I.

The material in Chapter IV concentrated on hypothesis testing problems. A detailed example of simultaneous hypothesis testing utilizing the Balanced Incomplete Block Design was examined to clarify the simultaneous testing procedure. Simultaneous confidence interval estimation was the subject of Chapter V. A number of procedures for obtaining approximate simultaneous confidence intervals were developed and the generality of the procedures affords solutions to a wide class of problems.

Perhaps one of the most important results of this treatment was mentioned in Chapter VI. Observing the tables presented in Appendix I of the bivariate Chi-square and bivariate F critical points, it was pointed out that except for extremely large values of p the univariate tables can provide excellent approximations to the bivariate critical points. In essence this means that unless the correlation between the bivariate random variables is very large the random variables may be treated as being independent (once transformations are made as in Chapter IV to remove confounding). In addition, if one wishes to estimate the critical points based on the correlation ρ , a formula which gives good estimates for a wide range of the correlation was presented.

Table A-1

Bivariate Density of Equation (3.1)

20	3	5	7	9	13	19	28	47	104
16	3	4	6	8	12	17	25	41	90
12	3	4	6	8	10	14	21	35	76
8	2	4	5	6	9	12	17	28	61
4	2	3	4	5	7	9	13	21	44
3	2	3	4	5	6	8	12	18	39
2	2	3	3	4	5	7	10	16	33
1	2	2	3	4	5	6	8	13	26

Table A-2

Upper 2 1/2% Critical Points of the Bivariate Chi-square
Distribution of Equation (3.1)

ρ m	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	6.23	6.22	6.20	6.18	6.14	6.09	6.00	5.89	5.69
2	8.75	8.74	8.73	8.71	8.67	8.62	8.53	8.40	8.17
3	10.85	10.85	10.83	10.80	10.77	10.71	10.62	10.48	10.23
4	12.75	12.75	12.73	12.70	12.67	12.61	12.51	12.36	12.11
8	19.46	19.46	19.44	19.41	19.37	19.32	19.22	19.03	18.73
12	25.50	25.50	25.49	25.46	25.42	25.35	25.25	25.05	24.70
16	31.23	31.22	31.20	31.18	31.13	31.07	30.94	30.74	30.35
20	36.74	36.73	36.71	36.68	36.64	36.58	36.44	36.22	35.80

Table A-3

Upper 5% Critical Points of the Bivariate Chi-square
Distribution of Equation (3.1)

$\rho \backslash m$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	5.00	4.99	4.97	4.94	4.89	4.84	4.75	4.63	4.45
2	7.35	7.34	7.32	7.29	7.24	7.18	7.09	6.95	6.73
3	9.32	9.31	9.29	9.26	9.21	9.14	9.05	8.90	8.65
4	11.11	11.10	11.08	11.05	11.00	10.93	10.83	10.67	10.40
8	17.50	17.48	17.46	17.43	17.38	17.30	17.18	16.99	16.66
12	23.29	23.28	23.26	23.22	23.17	23.08	22.95	22.73	22.37
16	28.80	28.78	28.76	28.72	28.66	28.57	28.43	28.20	27.79
20	34.12	34.10	34.08	34.04	33.98	33.88	33.73	33.47	33.03

Table A-4

Upper 10% Critical Points of the Bivariate Chi-square
Distribution of Equation (3.1)

$\rho \backslash m$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	3.79	3.78	3.76	3.72	3.67	3.61	3.52	3.41	3.23
2	5.94	5.92	5.90	5.86	5.81	5.73	5.64	5.50	5.28
3	7.75	7.74	7.71	7.67	7.62	7.54	7.43	7.27	7.03
4	9.42	9.41	9.38	9.34	9.28	9.20	9.08	8.91	8.64
8	15.42	15.41	15.38	15.33	15.26	15.17	15.03	14.82	14.48
12	20.93	20.91	20.88	20.83	20.75	20.64	20.49	20.25	19.86
16	26.19	26.17	26.14	26.08	26.00	25.88	25.71	25.45	25.01
20	31.30	31.28	31.24	31.18	31.10	30.97	30.78	30.49	30.02

Table A-5

Upper 5% Critical Points of the Bivariate Chi-square
Distribution of Equation (3.2)

m	$m+n$	$p = 0.2$			$p = 0.4$			$p = 0.6$		
		C_1	C_2	C_1	C_1	C_2	C_1	C_1	C_2	C_2
2	4	7.37	11.08	7.34	11.04	7.29	10.97	14.28	17.37	20.34
2	6	7.37	14.38	7.35	14.34	7.32	17.37	20.34	23.19	26.00
2	8	7.37	17.47	7.35	17.43	7.32	20.34	23.19	26.00	28.70
2	10	7.36	20.43	7.34	20.40	7.32	23.16	26.00	28.70	31.30
2	12	7.37	23.26	7.36	23.23	7.35	25.87	28.59	31.30	33.99
4	6	11.10	14.40	11.07	14.36	10.99	14.26	17.33	20.32	23.08
4	8	11.10	17.50	11.07	17.46	11.02	17.38	20.35	23.12	25.92
4	10	11.08	20.47	11.06	20.43	11.02	23.08	25.87	28.59	31.30
4	12	11.10	23.30	11.08	23.26	11.05	25.92	28.67	31.36	34.05
4	14	11.09	26.08	11.08	26.05	11.06	28.70	31.30	33.99	36.74
6	8	14.40	17.49	14.36	17.45	14.26	17.33	20.32	23.08	25.87
6	10	14.37	20.47	14.34	20.42	14.27	23.16	26.00	28.70	31.30
6	12	14.40	23.30	14.37	23.25	14.32	25.92	28.67	31.36	34.05
6	14	14.39	26.09	14.37	26.05	14.33	28.70	31.30	33.99	36.74
6	16	14.40	28.80	14.38	28.76	14.35	31.30	33.99	36.74	39.43
8	10	17.47	20.45	17.42	20.40	17.32	23.12	25.92	28.67	31.36
8	12	17.49	23.28	17.45	23.23	17.37	25.87	28.59	31.30	33.99
8	14	17.48	26.07	17.45	26.02	17.39	28.67	31.36	34.05	36.74
8	16	17.49	28.79	17.46	28.74	17.42	31.30	33.99	36.74	39.43
8	18	17.49	31.48	17.46	31.43	17.42	33.99	36.74	39.43	42.12
10	12	20.46	23.26	20.41	23.21	20.29	23.08	25.87	28.67	31.36
10	14	20.46	26.04	20.41	25.99	20.32	25.92	28.67	31.36	34.05
10	16	20.47	28.76	20.43	28.70	20.35	28.70	31.30	33.99	36.74
10	18	20.46	31.45	20.42	31.39	20.36	31.30	33.99	36.74	39.43
10	20	20.45	34.11	20.41	34.05	20.37	33.99	36.74	39.43	42.12

Table A-6

Upper 2 1/2% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p = 0.1$

m \ s								
	1	2	3	4	8	12	16	20
12	8.52	6.36	5.48	4.98	4.12	3.78	3.60	3.48
16	7.86	5.78	4.93	4.45	3.62	3.30	3.12	3.01
20	7.49	5.46	4.63	4.16	3.35	3.04	2.86	2.75
24	7.25	5.26	4.44	3.98	3.18	2.87	2.70	2.59
30	7.03	5.06	4.26	3.81	3.02	2.71	2.54	2.43
40	6.82	4.88	4.09	3.64	2.86	2.55	2.38	2.27

Table A-7

Upper 2 1/2% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p = 0.2$

m \ s								
	1	2	3	4	8	12	16	20
12	8.50	6.35	5.47	4.97	4.11	3.77	3.59	3.48
16	7.84	5.77	4.92	4.44	3.62	3.30	3.12	3.01
20	7.47	5.45	4.62	4.16	3.35	3.03	2.86	2.75
24	7.24	5.25	4.43	3.97	3.18	2.87	2.69	2.58
30	7.02	5.06	4.25	3.80	3.02	2.71	2.54	2.43
40	6.80	4.88	4.08	3.64	2.86	2.55	2.38	2.27

Table A-8

Upper 2 1/2% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p = 0.3$

s \ m										
	1	2	3	4	8	12	16	20		
12	8.46	6.33	5.45	4.95	4.10	3.77	3.59	3.48		
16	7.81	5.75	4.91	4.43	3.61	3.29	3.11	3.01		
20	7.44	5.44	4.61	4.15	3.34	3.03	2.86	2.75		
24	7.21	5.24	4.42	3.97	3.17	2.86	2.69	2.58		
30	7.00	5.05	4.25	3.79	3.01	2.70	2.53	2.43		
40	6.78	4.86	4.08	3.63	2.86	2.55	2.38	2.27		

Table A-9

Upper 2 1/2% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p = 0.4$

s \ m										
	1	2	3	4	8	12	16	20		
12	8.41	6.30	5.42	4.93	4.08	3.75	3.58	3.48		
16	7.76	5.73	4.89	4.41	3.60	3.28	3.11	3.01		
20	7.40	5.41	4.59	4.13	3.33	3.02	2.85	2.75		
24	7.18	5.22	4.41	3.95	3.16	2.85	2.69	2.58		
30	6.96	5.03	4.23	3.78	3.00	2.70	2.53	2.43		
40	6.75	4.85	4.06	3.62	2.85	2.55	2.38	2.27		

Table A-10

Upper 2 1/2% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p = 0.5$

s \ m								
	1	2	3	4	8	12	16	20
12	8.33	6.25	5.39	4.91	4.05	3.73	3.55	3.46
16	7.70	5.69	4.86	4.40	3.58	3.26	3.09	2.99
20	7.35	5.38	4.57	4.12	3.32	3.00	2.84	2.73
24	7.13	5.19	4.38	3.94	3.15	2.84	2.67	2.57
30	6.91	5.00	4.21	3.77	2.99	2.69	2.52	2.42
40	6.71	4.82	4.04	3.61	2.84	2.54	2.37	2.26

Table A-11

Upper 2 1/2% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p = 0.6$

s \ m								
	1	2	3	4	8	12	16	20
12	8.23	6.18	5.34	4.87	4.02	3.70	3.53	3.43
16	7.61	5.63	4.82	4.37	3.55	3.24	3.07	2.97
20	7.26	5.33	4.53	4.09	3.29	2.99	2.82	2.72
24	7.05	5.14	4.35	3.91	3.13	2.83	2.66	2.56
30	6.84	4.96	4.18	3.75	2.98	2.67	2.50	2.40
40	6.64	4.78	4.02	3.59	2.83	2.52	2.36	2.25

Table A-12

Upper 5% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p=0.1$

s \ m								
	1	2	3	4	8	12	16	20
12	6.45	4.99	4.37	4.01	3.38	3.12	2.98	2.88
16	6.04	4.62	4.01	3.66	3.04	2.79	2.66	2.57
20	5.81	4.41	3.81	3.46	2.85	2.61	2.47	2.39
24	5.66	4.28	3.68	3.34	2.73	2.49	2.36	2.27
30	5.52	4.15	3.56	3.22	2.62	2.38	2.24	2.15
40	5.38	4.02	3.44	3.10	2.51	2.26	2.13	2.04

Table A-13

Upper 5% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p = 0.2$

s \ m								
	1	2	3	4	8	12	16	20
12	6.43	4.98	4.36	4.00	3.37	3.12	2.97	2.88
16	6.02	4.61	4.00	3.65	3.04	2.79	2.65	2.56
20	5.79	4.40	3.80	3.46	2.85	2.61	2.47	2.38
24	5.65	4.27	3.67	3.33	2.73	2.49	2.35	2.26
30	5.51	4.14	3.55	3.21	2.62	2.37	2.24	2.15
40	5.37	4.02	3.43	3.10	2.50	2.26	2.13	2.04

Table A-14

Upper 5% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p = 0.3$

$s \backslash m$	1	2	3	4	8	12	16	20
12	6.39	4.96	4.34	3.98	3.36	3.11	2.97	2.87
16	5.99	4.59	3.98	3.64	3.03	2.78	2.65	2.56
20	5.76	4.38	3.79	3.45	2.84	2.60	2.47	2.38
24	5.62	4.25	3.66	3.32	2.72	2.48	2.35	2.26
30	5.48	4.12	3.54	3.20	2.61	2.37	2.23	2.15
40	5.35	4.00	3.42	3.09	2.50	2.26	2.12	2.04

Table A-15

Upper 5% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p = 0.4$

$s \backslash m$	1	2	3	4	8	12	16	20
12	6.34	4.92	4.31	3.96	3.35	3.09	2.96	2.87
16	5.94	4.56	3.96	3.62	3.02	2.77	2.64	2.56
20	5.72	4.36	3.77	3.43	2.83	2.59	2.46	2.38
24	5.58	4.23	3.64	3.31	2.72	2.48	2.34	2.26
30	5.44	4.10	3.52	3.19	2.60	2.36	2.23	2.15
40	5.31	3.98	3.41	3.08	2.49	2.25	2.12	2.03

Table A-16

Upper 5% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $\rho = 0.5$

m \ s								
	1	2	3	4	8	12	16	20
12	6.26	4.88	4.28	3.94	3.32	3.08	2.94	2.85
16	5.88	4.52	3.93	3.60	3.00	2.76	2.62	2.54
20	5.66	4.32	3.74	3.41	2.82	2.58	2.45	2.36
24	5.52	4.19	3.62	3.29	2.70	2.46	2.33	2.25
30	5.39	4.07	3.50	3.17	2.59	2.35	2.22	2.14
40	5.26	3.95	3.39	3.06	2.48	2.24	2.11	2.02

Table A-17

Upper 5% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $\rho = 0.6$

m \ s								
	1	2	3	4	8	12	16	20
12	6.16	4.82	4.23	3.90	3.30	3.05	2.92	2.83
16	5.79	4.47	3.89	3.56	2.98	2.74	2.61	2.52
20	5.58	4.27	3.70	3.38	2.80	2.56	2.43	2.35
24	5.44	4.15	3.59	3.26	2.68	2.45	2.32	2.23
30	5.31	4.03	3.47	3.15	2.57	2.34	2.21	2.12
40	5.19	3.91	3.36	3.04	2.47	2.23	2.10	2.01

Table A-18

Upper 10% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p = 0.1$

m \ s								
	1	2	3	4	8	12	16	20
12	4.61	3.76	3.37	3.13	2.70	2.52	2.42	2.36
16	4.39	3.54	3.15	2.92	2.49	2.32	2.22	2.15
20	4.26	3.42	3.03	2.80	2.37	2.20	2.10	2.03
24	4.18	3.34	2.95	2.72	2.30	2.12	2.02	1.96
30	4.10	3.26	2.87	2.64	2.22	2.04	1.94	1.88
40	4.02	3.18	2.80	2.57	2.15	1.97	1.87	1.80

Table A-19

Upper 10% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p = 0.2$

m \ s								
	1	2	3	4	8	12	16	20
12	4.59	3.75	3.36	3.12	2.70	2.52	2.42	2.35
16	4.37	3.53	3.14	2.91	2.49	2.31	2.21	2.15
20	4.24	3.41	3.02	2.79	2.37	2.20	2.10	2.03
24	4.16	3.33	2.94	2.71	2.29	2.12	2.02	1.95
30	4.08	3.25	2.87	2.64	2.22	2.04	1.94	1.88
40	4.00	3.18	2.79	2.56	2.14	1.97	1.87	1.80

Table A-20

Upper 10% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p = 0.3$

s \ m								
	1	2	3	4	8	12	16	20
12	4.56	3.73	3.34	3.11	2.69	2.51	2.41	2.35
16	4.34	3.51	3.13	2.90	2.48	2.31	2.21	2.14
20	4.21	3.39	3.01	2.78	2.36	2.19	2.09	2.03
24	4.13	3.31	2.93	2.70	2.29	2.11	2.01	1.95
30	4.05	3.23	2.85	2.63	2.21	2.04	1.94	1.87
40	3.97	3.16	2.78	2.55	2.14	1.96	1.86	1.80

Table A-21

Upper 10% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p = 0.4$

s \ m								
	1	2	3	4	8	12	16	20
12	4.51	3.70	3.31	3.09	2.67	2.50	2.40	2.34
16	4.29	3.48	3.11	2.88	2.47	2.30	2.20	2.14
20	4.17	3.36	2.99	2.76	2.35	2.18	2.08	2.02
24	4.09	3.29	2.91	2.69	2.28	2.10	2.01	1.94
30	4.01	3.21	2.84	2.61	2.20	2.03	1.93	1.87
40	3.94	3.14	2.76	2.54	2.13	1.95	1.86	1.79

Table A-22

Upper 10% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p = 0.5$

m \ s								
	1	2	3	4	8	12	16	20
12	4.44	3.65	3.28	3.06	2.65	2.48	2.39	2.33
16	4.23	3.45	3.08	2.86	2.45	2.28	2.19	2.13
20	4.11	3.33	2.96	2.74	2.34	2.17	2.07	2.01
24	4.03	3.25	2.89	2.67	2.26	2.09	2.00	1.93
30	3.96	3.18	2.81	2.59	2.19	2.02	1.92	1.86
40	3.88	3.11	2.74	2.52	2.12	1.94	1.85	1.78

Table A-23

Upper 10% Critical Points of the Bivariate F
Distribution of Equation (6.4) with $p = 0.6$

m \ s								
	1	2	3	4	8	12	16	20
12	4.35	3.60	3.24	3.02	2.63	2.46	2.37	2.31
16	4.15	3.39	3.04	2.82	2.43	2.26	2.17	2.11
20	4.03	3.28	2.92	2.71	2.32	2.15	2.06	2.00
24	3.96	3.21	2.85	2.64	2.24	2.08	1.98	1.92
30	3.88	3.14	2.78	2.57	2.17	2.00	1.91	1.85
40	3.81	3.07	2.71	2.50	2.10	1.93	1.83	1.77

APPENDIX 2

In Chapter V confidence bounds were derived contingent on

the condition that

$$\Pr \left\{ s_1 < c, s_2 < c \right\} \text{ and } \Pr \left\{ s_1 > c, s_2 > c \right\},$$

where (s_1, s_2) is a bivariate Chi-square random variable, is a mono-

tonic increasing function of ρ^2 . This is equivalent to the condition

that these probabilities are monotonic increasing functions of ρ for

$0 < \rho < 1$. The proof of this conjecture follows.

$$\text{Note that } \Pr \left\{ s_1 < c, s_2 < c \right\} \text{ is equivalent to}$$

$$\Pr \left\{ x_1^2 + \dots + x_m^2 < c, y_1^2 + \dots + y_m^2 < c \right\}$$

where (x_i, y_i) are independent bivariate normal random variables

with correlation ρ . Only the equicorrelated case need be considered

here since the general case is handled by using an approximate equi-

correlated bivariate Chi-square distribution

$$\rho^2 = \frac{1}{k} \sum_{j=1}^k \rho_j^2 \quad (\text{with } \rho_j^2 = \rho^2)$$

In addition, since the correlation ρ appears in the bivariate Chi-

square density only as a function of ρ^2 , the only interval one must

consider is $0 < \rho < 1$. Then,

$$\frac{\partial^2 f(x_1, y_1)}{\partial x_1 \partial y_1} = \frac{\partial^2 f(x_1, y_1)}{\partial x_1 \partial y_1} \quad i = 1, 2, \dots, m.$$

and by direct differentiation it is easily verified that

$$f(x_1, y_1) = \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[x_1^2 - 2\rho x_1 y_1 + y_1^2 \right] \right\}$$

1, 2, ..., m. The density function of (x_1, y_1) is

where the subscripts $k_1, k_2, \dots, k_m$ can be any permutation of

$$P = \Pr \left\{ s_1 < c, s_2 < c \right\} = \Pr \left\{ x_1^2 + x_2^2 + \dots + x_m^2 < c \right\} = \Pr \left\{ x_{k_1}^2 < c, x_{k_2}^2 < c - x_{k_1}^2, \dots, x_{k_m}^2 < c - x_{k_1}^2 - x_{k_2}^2 - \dots - x_{k_{m-1}}^2 \right\} < c - \sum_{j \neq km} y_j^2 < c - \sum_{j \neq km} y_j^2$$

Let

$$a_{k1} = \sqrt{c^2 - \sum_{j \neq km} x_{kj}^2}, \dots, a_{k2} = \sqrt{c^2 - \sum_{j \neq km} x_{kj}^2}$$

and

$$b_{k1} = \sqrt{c^2 - \sum_{j \neq km} y_{kj}^2}, \dots, b_{k2} = \sqrt{c^2 - \sum_{j \neq km} y_{kj}^2}$$

Then,

$$P = \prod_{j=1}^m \int_{b_{kj}}^{-b_{kj}} \int_{a_{kj}}^{-a_{kj}} f(x_{kj}, y_{kj}) dx_{kj} dy_{kj}.$$

By appropriate choice of the subscripts $k1, k2, \dots, km$ in each of

the summations below, $\frac{\partial P}{\partial p}$ can be written as

$$\frac{\partial P}{\partial p} = \sum_{j=1}^m \prod_{i \neq j} \int_{b_{ki}}^{-b_{ki}} \int_{a_{ki}}^{-a_{ki}} f(x_{ki}, y_{ki})$$

$$* dx_{ki} dy_{ki} \left\{ \frac{\partial}{\partial p} \int_{b_{kj}}^{-b_{kj}} \int_{a_{kj}}^{-a_{kj}} \right.$$

$$* f(x_{kj}, y_{kj}) dx_{kj} dy_{kj} \left. \right\}.$$

But,

$$\frac{\partial}{\partial p} \int_{b_{kj}}^{-b_{kj}} \int_{a_{kj}}^{-a_{kj}} f(x_{kj}, y_{kj}) dx_{kj} dy_{kj} = \int_{b_{kj}}^{-b_{kj}} \int_{a_{kj}}^{-a_{kj}}$$

$$f(a_{kj}, b_{kj}) - f(a_{kj}, -b_{kj}) - f(-a_{kj}, b_{kj}) + f(-a_{kj}, -b_{kj}) \geq 0$$

neither a_{kj} nor b_{kj} are zero. Thus

For $0 < \rho < 1$, (A-2.1) is strictly greater than (A-2.2) provided

$$e^* - \frac{1}{2(1-\rho^2)} \left[a_{kj}^2 + 2\rho b_{kj} a_{kj} + b_{kj}^2 \right] \quad (A-2.2)$$

$$f(-a_{kj}, b_{kj}) = f(a_{kj}, -b_{kj}) = \frac{1}{I} \sqrt{1-\rho^2} e^{2\pi i \rho}$$

and

$$e^* - \frac{1}{2(1-\rho^2)} \left[a_{kj}^2 - 2\rho b_{kj} a_{kj} + b_{kj}^2 \right] \quad (A-2.1)$$

$$\text{Now } f(a_{kj}, b_{kj}) = f(-a_{kj}, -b_{kj}) = \frac{1}{I} \sqrt{1-\rho^2} e^{2\pi i \rho}$$

$$= f(a_{kj}, b_{kj}) - f(a_{kj}, -b_{kj}) - f(-a_{kj}, b_{kj}) + f(-a_{kj}, -b_{kj})$$

$$= \int_{a_{kj}}^{b_{kj}} \int_{-a_{kj}}^{-b_{kj}} \frac{\partial^2 f(x_{kj}, y_{kj})}{\partial x_{kj} \partial y_{kj}} dx_{kj} dy_{kj}$$

and since A i

$$\int_{a_{ki}}^{b_{ki}} \int_{a_{ki}}^{b_{ki}} f(x_{ki}, y_{ki}) dx_{ki} dy_{ki} > 0$$

for all b_{ki} and a_{ki} not equal to zero, it follows that $\frac{\partial P}{\partial p} > 0$ for

$0 < p < 1$. Hence P is a monotone increasing function of p^2 .

A similar argument can be used to show that

$$\Pr \left\{ s_1 > c, s_2 > c \right\}$$

is a monotone increasing function of p^2 , and corresponding argu-

ments hold for the bivariate F distribution since probabilities are

obtained from the bivariate Chi-square density function.

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13. ABSTRACT

Frequently the experimenter is confronted with an analysis involving two or more Chi-square random variables. This is often the case, for example, in experimental design or regression problems. Provided the Chi-square random variables are independent, probabilities associated with the random variables are easily determined. If the Chi-square random variables are not independent the analyses are much more complex and few practical tools are currently available to the experimenter. In this work the case of two dependent Chi-square random variables is discussed. A unified approach to solving the distributional problems is presented using canonical analysis. Density functions are derived for the bivariate Chi-square random variable with a special type of dependence. From a study of moments of the distribution, an approximation for any type of dependence using the special form mentioned above is suggested. The forms of the bivariate Chi-square density function derived in this paper are especially useful because they can be easily programmed on a computer. Thus, probabilities and critical points can be determined. Applications of the bivariate Chi-square distributions are also presented in this study. In particular simultaneous testing problems associated with the balanced incomplete Block Design are discussed. A number of results relating to simultaneous confidence interval estimation are also presented. Finally, the computation and estimation of critical points of the bivariate Chi-square and the related bivariate F distributions are examined.